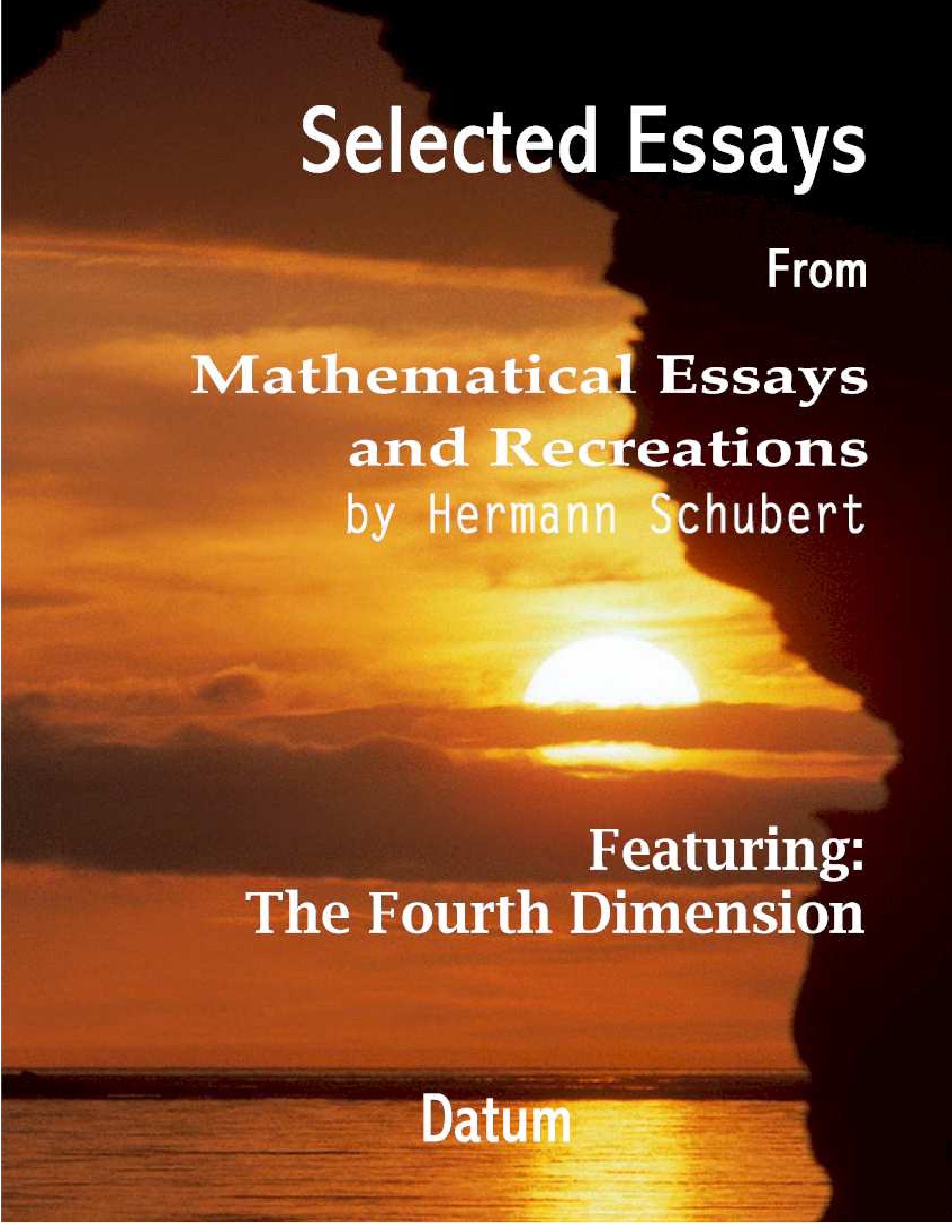

Selected Essays

From

Mathematical Essays and Recreations

by Hermann Schubert

Featuring: The Fourth Dimension

Datum

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Featuring:
The Fourth Dimension

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1898

This electronic version by
Datum, Publisher

http://4DLab.info

January 25, 2009

Book layout and design by E. Pérez

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Translator's Note

THE MATHEMATICAL ESSAYS and recreations in this volume are by one of the most successful teachers and text-book writers of Germany. The monistic construction of arithmetic, the systematic and organic development of all its consequences from a few thoroughly established principles, is quite foreign to the general run of American and English elementary text-books, and the first three essays of Professor Schubert will, therefore, from a logical and esthetic side, be full of suggestions for elementary mathematical teachers and students, as well as for non-mathematical readers. For the actual detailed development of the system of arithmetic here sketched, we may refer the reader to Professor Schubert's volume *Arithmetik und Algebra*, recently published in the Göschen-Sammlung (Göschen, Leipsic),—an extraordinarily cheap series containing many other unique and valuable text-books in mathematics and the sciences.

The remaining essays on "Magic Squares," "The Fourth Dimension," and "The History of the Squaring of the Circle," will be found to be the most complete generally accessible accounts in English, and to have, one and all, a distinct educational and ethical lesson.

In all these essays, which are of a simple and popular character, and designed for the general public, Professor Schubert has incorporated much of his original research.

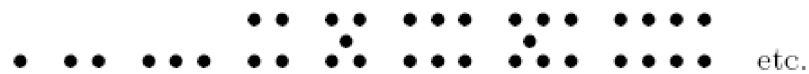
Thomas J. McCormack. La Salle, Ill., December, 1898.

Notion and Definition of Number

MANY ESSAYS HAVE BEEN WRITTEN on the definition of number. But most of them contain too many technical expressions, both philosophical and mathematical, to suit the non-mathematician. The clearest idea of what counting and numbers mean may be gained from the observation of children and of nations in the childhood of civilisation. When children count or add, they use either their fingers, or small sticks of wood, or pebbles, or similar things, which they adjoin singly to the things to be counted or otherwise ordinally associate with them. As we know from history, the Romans and Greeks employed their fingers when they counted or added. And even to-day we frequently meet with people to whom the use of the fingers is absolutely indispensable for computation.

Still better proof that the accurate association of such "other" things with the things to be counted is the essential element of numeration are the tales of travellers in Africa, telling us how African tribes sometimes inform friendly nations of the number of the enemies who have invaded their domain. The conveyance of the information is effected not by messengers, but simply by placing at spots selected for the purpose a number of stones exactly equal to the number of the invaders. No one will deny that the number of the tribe's foes is thus communicated, even though no name exists for this number in the languages of the tribes. The reason why the fingers are so universally employed as a means of numeration is, that every one possesses a definite number of fingers, sufficiently large for purposes of computation and that they are always at hand.

Besides this first and chief element of numeration which, as we have seen, is the exact, individual conjunction or association of other things with the things to be counted, is to be mentioned a second important element, which in some respects perhaps is not so absolutely essential; namely, that the things to be counted shall be regarded as of the same kind. Thus, any one who subjects apples and nuts collectively to a process of numeration will regard them for the time being as objects of the same kind, perhaps by subsuming them under the common notion of fruit. We may therefore lay down provisionally the following as a definition of counting: to count a group of things is to regard the things as the same in kind and to associate ordinally, accurately, and singly with them other things. In writing, we associate with the things to be counted simple signs, like points, strokes, or circles. The form of the symbols we use is indifferent. Neither need they be uniform. It is also indifferent what the spatial relations or dispositions of these symbols are. Although, of course, it is much more convenient and simpler to fashion symbols growing out of operations of counting on principles of uniformity and to place them spatially near each other. In this manner are produced what I have called[1] natural number-pictures; for example,



Now-a-days such natural number-pictures are rarely employed, and are to be seen only on dominoes, dice, and sometimes, also, on playingcards.

It can be shown by archæological evidence that originally numeral writing was made up wholly of natural number-pictures. For example, the Romans in early times represented all numbers, which were written at all, by assemblages of strokes. We have remnants of this writing in the first three numerals of the modern Roman system. If we needed additional evidence that the Romans originally employed natural number-signs, we might cite the passage in Livy, VII. 3, where we are told, that, in accordance with a very ancient law, a nail was annually driven into a certain spot in the sanctuary of Minerva, the "inventrix" of counting, for the purpose of showing the number of years which had elapsed since the building of the edifice. We learn from the same source that also in the temple at Volsinii nails were shown which the Etruscans had placed there as marks for the number of years.

Also recent researches in the civilisation of ancient Mexico show that natural number-pictures were the first stage of numeral notation. Whosoever has carefully studied in any large ethnographical collection the monuments of ancient Mexico, will surely have remarked that the nations which inhabited Mexico before its conquest by the Spaniards, possessed natural number-signs for all numbers from one to nineteen, which they formed by combinations of circles. If in our studies of the past of modern civilised peoples, we meet with natural number-pictures only among the Greeks or Romans, and some Oriental nations, the reason is that the other nations, as the Germans, before they

[1]*System der Arithmetik.* (Potsdam: Aug. Stein. 1885.)

came into contact with the Romans and adopted the more highly developed notation of the latter, were not yet sufficiently advanced in civilisation to feel any need of expressing numbers symbolically. But since the most perfect of all systems of numeration, the Hindu system of "local value," was introduced and adopted in Europe in the twelfth century, the Roman numeral system gradually disappeared, at least from practical computation, and at present we are only reminded by the Roman characters of inscriptions of the first and primitive stage of all numeral notation. To-day we see natural number-pictures, except in the abovementioned games, only very rarely, as where the tally-men of wharves or warehouses make single strokes with a pencil or a piece of chalk, one for each bale or sack which is counted.

As in writing it is of consequence to associate with each of the things to be counted some simple sign, so in speaking it is of consequence to utter for each single thing counted some short sound. It is quite indifferent here what this sound is called, also whether the sounds which are associated with the things to be counted are the same in kind or not, and finally, whether they are uttered at equal or unequal intervals of time. Yet it is more convenient and simpler to employ the same sound and to observe equal intervals in their utterance. We arrive thus at natural number-words. For example, utterances like,

oh, oh-oh, oh-oh-oh, oh-oh-oh-oh, oh-oh-oh-oh-oh,

are natural number-words for the numbers from one to five. Numberwords of this description are not now to be found in any known language. And yet we hear such natural number-words constantly, every day and night of our lives; the only difference being that the speakers are not human beings but machines—namely, the striking-apparatus of our clocks.

Word-forms of the kind described are too inconvenient, however, for use in language, not only for the speaker, on account of their ultimate length, but also for the hearer, who must be constantly on the qui vive lest he misunderstand a numeral word so formed. It has thus come about that the languages of men from time immemorial have possessed numeral words which exhibit no trace of the original idea of single association. But if we should always select for every new numeral word some new and special verbal root, we should find ourselves in possession of an inordinately large number of roots, and too severely tax our powers of memory. Accordingly, the languages of both civilised and uncivilised peoples always construct their words for larger numbers from words for smaller numbers. What number we shall begin with in the formation of compound numeral words is quite indifferent, so far as the idea of number itself is concerned. Yet we find, nevertheless, in nearly all languages one and the same number taken as the first station in the formation of compound numeral words, and this number is ten. Chinese and Latins, Fins and Malays, that is, peoples who have no linguistic relationship, all display in the formation of numeral words the similarity of beginning with the number ten the formation of compound numerals. No other reason can be found for this striking agreement than the fact that all the forefathers of these nations possessed ten fingers.

Granting it were impossible to prove in any other way that people originally used their fingers in reckoning, the conclusion could be inferred with sufficient certainty solely from this agreement with regard to the first resting-point in the formation of compound numerals among the most various races. In the Indo-Germanic tongues the numeral words from ten to ninety-nine are formed by composition from smaller numeral words. Two methods remain for continuing the formation of the numerals: either to take a new root as our basis of composition (hundred) or to go on counting from ninety-nine, saying tenty, eleventy, etc. If we were logically to follow out this second method we should get tenty-ty for a thousand, tenty-ty-ty for ten thousand, etc. But in the utterance of such words, the syllable ty would be so frequently repeated that the same inconvenience would be produced as above in our individual number-pictures. For this reason the genius which controls the formation of speech took the first course.

But this course is only logically carried out in the old Indian numeral words. In Sanskrit we not only have for ten, hundred, and thousand a new root, but new bases of composition also exist for ten thousand, one hundred thousand, ten millions, etc., which are in no wise related with the words for smaller numbers. Such roots exist among the Hindus for all numerals up to the number expressed by a one and fifty-four appended naughts. In no other language do we find this principle carried so far. In most languages the numeral words for the numbers consisting of a one with four and five appended naughts are compounded, and in further formations use is made of the words million, billion, trillion, etc., which really exhibit only one root, before which numeral words of the Latin tongue are placed.

Besides numeral word-systems based on the number ten, only logical systems are found based on the number five and on the number twenty. Systems of numeral words which have the basis five occur in equatorial Africa. (See the language-tables of Stanley's books on Africa.) The Aztecs and Mayas of ancient Mexico had the base twenty. In Europe it was mainly the Celts who reckoned with twenty as base. The French language still shows some few traces of the Celtic vicenary system, as in its word for eighty, *quatre-vingt*. The choice of five and of twenty as bases is explained simply enough by the fact that each hand has five fingers, and that hands and feet together have twenty fingers and toes.

As we see, the languages of humanity now no longer possess natural number-signs and number-words, but employ names and systems of notation adopted subsequently to this first stage. Accordingly, we must add to the definition of counting above given a third factor or element which, though not absolutely necessary, is yet important, namely, that we must be able to express the results of the above-defined associating of certain other things with the things to be counted, by some conventional sign or numeral word.

Having thus established what counting or *numbering* means, we are in a position to define also the notion of *number*, which we do by simply saying that by number we understand *the results* of counting or numeration. These are naturally composed of two elements. First, of the ordinary number-word or number-sign; and secondly, of the word standing for the specific things counted. For example, eight men, seven trees, five cities. When, now, we have counted one group of things, and subsequently also counted another group of things of the same kind, and thereupon we conceive the two groups of things combined into a single group, we can save ourselves the labor of counting the things a third time by blending the number-pictures belonging to the two groups into a single number-picture belonging to the whole. In this way we arrive on the one hand at the idea of addition, and on the other, at the notion of "unnamed" number. Since we have no means of telling from the two original number-pictures and the third one which is produced from these, the kind or character of the things counted, we are ultimately led in our conception of number to abstract wholly from the nature of the things counted, and to form the definition of unnamed number.

We thus see that to ascend from the notion of named number to the notion of unnamed number, the notion of addition, joined to a high power of abstraction, is necessary. Here again our theory is best verified by observations of children learning to count and add. A child, in beginning arithmetic, can well understand what five pens or five chairs are, but he cannot be made to understand from this alone what five abstractly is. But if we put beside the first five pens three other pens, or beside the five chairs three other chairs, we can usually bring the child to see that five things plus three things are always eight things, no matter of what nature the things are, and that accordingly we need not always specify in counting what kind of things we mean. At first we always make the answer to our question of what five plus three is, easy for the child, by relieving him of the process of abstraction, which is necessary to ascend from the named to the unnamed number, an end which we accomplish by not asking first what five plus three is, but by associating with the numbers words designating things within the sphere of the child's experience, for example, by asking how many five pens plus three pens are.

The preceding reflexions have led us to the notion of unnamed or abstract numbers. The arithmetician calls these numbers positive whole numbers, or positive integers, as he knows of other kinds of numbers, for example, negative numbers, irrational numbers, etc. Still, observation of the world of actual facts, as revealed to us by our senses, can naturally lead us only to positive whole numbers, such only, and no others, being results of actual counting. All other kinds of numbers are nothing but artificial inventions of mathematicians created for the purpose of giving to the chief tool of the mathematician, namely, arithmetical notation, a more convenient and more practical form, so that the solution of the problems which arise in mathematics may be simplified. All numbers, excepting the results of counting above defined, are and remain mere symbols, which, although they are of incalculable value in mathematics, and, therefore, can scarcely be dispensed with, yet could, if it were a question of principle, be avoided. Kronecker has shown that any problem in which positive whole numbers are given, and only such are sought, always admits of solution without the help of other kinds of numbers, although the employment of the latter wonderfully simplifies the solution.

How these derived species of numbers, by the logical application of a single principle, flow naturally from the notion of number and of addition above deduced, I shall show in the next article entitled ”Monism in Arithmetic.”

Monism in Arithmetic

In his *Primer of Philosophy*, Dr. Paul Carus defines monism as a "unitary conception of the world." Similarly, we shall understand by monism in a science the unitary conception of that science. The more a science advances the more does monism dominate it. An example of this is furnished by physics. Whereas formerly physics was made up of wholly isolated branches, like Mechanics, Heat, Optics, Electricity, and so forth, each of which received independent explanations, physics has now donned an almost absolute monistic form, by the reduction of all phenomena to the *motions* of molecules. For example, optical and electrical phenomena, we now know, are caused by the undulatory movements of the ether, and the length of the ether-waves constitutes the sole difference between light and electricity.

Still more distinctly than in physics is the monistic element displayed in pure arithmetic, by which we understand the theory of the combination of two numbers into a third by addition and the direct and indirect operations springing out of addition. Pure arithmetic is a science which has completely attained its goal, and which can prove that it has, exclusively by internal evidence. For it may be shown on the one hand that besides the seven familiar operations of addition, subtraction, multiplication, division, involution, evolution, and the finding of logarithms, no other operations are definable which present anything essentially new; and on the other hand that fresh extensions of the domain of numbers beyond irrational, imaginary, and complex numbers are arithmetically impossible. Arithmetic may be compared to a tree that has completed its growth, the boughs and branches of which may still increase in size or even give forth fresh sprouts, but whose main trunk has attained its fullest development.

Since arithmetic has arrived at its maturity, the more profound investigation of the nature of numbers and their combinations shows that a unitary conception of arithmetic is not only possible but also necessary. If we logically abide by this unitary conception, we arrive, starting from the notion of counting and the allied notion of addition, at all conceivable operations and at all possible extensions of the notion of number. Although previously expressed by Grassmann, Hankel, E. Schröder, and Kronecker, the author of the present article, in his "System of Arithmetic," Potsdam, 1885, was the first to work out the idea referred to, fully and logically and in a form comprehensible for beginners. This book, which Kronecker in his "Notion of Number," an essay published in Zeller's jubilee work, makes special mention of, is intended for persons proposing to learn arithmetic. As that cannot be the object of the readers of these essays, whose purpose will rather be the study of the logical construction of the science from some single fundamental principle, the following pages will simply give of the notions and laws of arithmetic what is absolutely necessary for an understanding of its development.

The starting-point of arithmetic is the idea of counting and of number as the result of counting. On this subject, the reader is requested to read the first essay of this collection. It is there shown that the idea of addition springs immediately from the idea of counting. As in counting it is indifferent in what order we count, so in addition it is indifferent, for the sum, or the result of the addition, whether we add the first number to the second or the second to the first. This law, which in the symbolic language of arithmetic, is expressed by the formula

$$a + b = b + a,$$

is called the *commutative law* of addition. Notwithstanding this law, however, it is evidently desirable to distinguish the two quantities which are to be summed, and out of which the sum is produced, by special names. As a fact, the two summands usually are distinguished in some way, for example, by saying a is to be increased by b, or b is to be added to a, and so forth. Here, it is plain, a is always something that is to be increased, b the increase. Accordingly it has been proposed to call the number which is regarded in addition as the passive number or the one to be changed, the *augend*, and the other which plays the active part, which accomplishes the change, so to speak, the *increment*. Both words are derived from the Latin and are appropriately chosen. Augend is derived from *augere*, to increase, and signifies that which is to be increased; increment comes from *increscere*, to grow, and signifies as in its ordinary meaning what is added.

Besides the commutative law one other follows from the idea of counting—the *associative law* of addition. This law, which has reference not to two but to three numbers, states that having a certain sum, $a + b$, it is indifferent for the result whether we increase the increment b of that sum by a number, or whether we increase the sum itself by the same number. Expressed in the symbolic language of arithmetic this law reads,

$$a+(b+c)=(a+b)+c.$$

To obtain now all the rules of addition we have only to apply the two laws of commutation and association above stated, though frequently, in the deduction of the same rule, each must be applied many times. I may pass over here both the rules and their establishment.

In addition, two numbers, the augend a and the increment b are combined into a third number c, the sum. From this operation spring necessarily two inverse operations, the common feature of which is, that the sum sought in addition is regarded in both as known, and the difference that in the one the augend also is regarded as known, and in the other the increment. If we ask what number added to a gives c, we seek the increment. If we ask what number increased by b gives c, we seek the augend. As a matter of reckoning, the solution of the two questions is the same, since by the commutative law of addition $a+b=b+a$. Consequently, only one common name is in use for the two inverses of addition, namely, *subtraction*. But with respect to the notions involved, the two operations do differ, and it is accordingly desirable in a logical investigation of the structure of arithmetic, to distinguish the two by different names. As in all probability no terms have yet been suggested for these two kinds of subtraction, I propose here for the first time the following words for the two operations, namely, *detraction* to denote the finding of the increment, and *subtertraction* to denote the finding of the augend. We obtain these terms simply enough by thinking of the augmentation of some object already existing. For example, the cathedral at Cologne had in its tower an augend that waited centuries for its increment, which was only supplied a few decades ago. As the cathedral had originally a height of one hundred and thirty metres, but after completion was increased in height twenty-six metres, of the total height of one hundred and fifty-six metres one hundred and thirty metres is clearly the augend and twenty-six metres the increment. If, now, we wished to recover the augend we should have to pull down (Latin, *detrahere*) the upper part along the whole height. Accordingly, the finding of the augend is called *detraction*. If we sought the increment, we should have to pull out the original part from beneath (Latin, *subtertrahere*). For this reason, the finding of the increment is called *subtertraction*. Owing to the commutative law, the two inverse operations, as matters of computation, become one, which bears the name of *subtraction*. The sign of this operation is the minus sign, a horizontal stroke. The number which originally was sum, is called in subtraction minuend; the number which in addition was increment is now called detractor; the number which in addition was augend is now called subtertractor. Comprising the two conceptually different operations in one single operation, subtraction, we employ for the number which before was increment or augend, the term subtrahend, a word which on account of its passive ending is not very good, and for which, accordingly, E. Schröder proposes to substitute the word *subtrahent*, having an active ending. The result of subtraction, or what is the same thing, the number sought, is called the *difference*. The definition-formula of subtraction reads

$$a - b + b = a,$$

that is, *a* minus *b* is the number which increased by *b* gives *a*, or the number which added to *b* gives *a*, according as the one or the other of the two operations inverse to addition is meant. From the formula for subtraction, and from the rules which hold for addition, follow now at once the rules which refer to both addition and subtraction. These rules we here omit.

From the foregoing it is plain that the minuend is necessarily larger than the subtrahent. For in the process of addition the minuend was the sum, and the sum grew out of the union of two natural number-pictures.[2] Thus 5 minus 9, or 11 minus 12, or 8 minus 8, are combinations of numbers *wholly destitute of meaning*; for no number, that is, no result of counting, exists that added to 9 gives the sum 5, or added to 12 gives the sum 11, or added to 8 gives 8. What, then, is to be done? Shall we banish entirely from arithmetic such meaningless combinations of numbers; or, since they have no meaning, shall we rather invest them with one? If we do the first, arithmetic will still be confined in the strait-jacket into which it was forced by the original definition of number as the result of counting. If we adopt the latter alternative we are forced to extend our notion of number. But in doing this, we sow the first seeds of the science of pure arithmetic, an organic body of knowledge which fructifies all other provinces of science.

What significance, then, shall we impart to the symbol

$$5 - 9?$$

Since 5 minus 9 possesses no significance whatever, we may, of course, impart to it any significance we wish. But as a matter of practical convenience it should be invested with no meaning that is likely to render it subject to exceptions. As the form of the symbol $5 - 9$ is the form of a difference, it will be obviously convenient to give it a meaning which will allow us to reckon with it as we reckon with every other real difference, that is, with a difference in which the minuend is larger than the subtrahent. This being agreed upon, it follows at once that all such symbols in which the number before the minus sign is less

[2] See page 4, *supra*.

than the number behind it by the same amount may be put equal to one another. It is practical, therefore, to comprise all these symbols under some one single symbol, and to construct this latter symbol so that it will appear unequivocally from it by how much the number before the minus sign is less than the number behind it. This difference, accordingly, is written down and the minus sign placed before it.

If the two numbers of such a differential *form* are equal, a totally new sign must be invented for the expression of the fact, having no relation to the signs which state results of counting. This invention was not made by the ancient Greeks, as one might naturally suppose from the high mathematical attainments of that people, but by Hindu Brahman priests at the end of the fourth century after Christ. The symbol which they invented they called *tsiphra*, empty, whence is derived the English *cipher*. The form of this sign has been different in different times and with different peoples. But for the last two or three centuries, since the symbolic language of arithmetic has become thoroughly established as an international character, the form of the sign has been 0 (French *zero*, German *null*).

In calling this symbol and the symbols formed of a minus sign followed by a result of counting, *numbers*, we widen the province of numbers, which before was wholly limited to results of counting. In no other way can zero and the negative numbers be introduced into arithmetic. No man can prove that 7 minus 11 is equal to 1 minus 5. Originally, both are meaningless symbols. And not until we agree to impart to them a significance which allows us to reckon with them as we reckon with real differences are we led to a statement of identity between 7 minus 11 and 1 minus 5. It was a long time before the negative numbers mentioned acquired the full rights of citizenship in arithmetic. Cardan called them, in his *ArsMagna*, 1545, *numeri ficti* (imaginary numbers), as distinguished from numeri veri (real numbers). Not until Descartes, in the first half of the seventeenth century, was any one bold enough to substitute *numeri ficti* and *numeriveri* indiscriminately for the same letter of algebraic expressions.

We have invested, thus, combinations of signs originally meaningless, in which a smaller number stood before than after a minus sign, with a meaning which enables us to reckon with such *apparent* differences exactly as we do with ordinary differences. Now it is just this practical shift of imparting meanings to combinations, which logically applied deduces naturally the whole system of arithmetic from the idea of counting and of addition, and which we may characterise, therefore, as the *foundation-principle* of its whole construction. This principle, which Hankel once called the *principle of permanence*, but which I prefer to call the PRINCIPLE OF NO EXCEPTION, may be stated in general terms as follows:

In the construction of arithmetic every combination of two previously defined numbers by a sign for a previously defined operation (plus, minus, times, etc.) shall be invested with meaning, even where the original definition of the operation used excludes such a combination; and the meaning imparted is to be such that the combination considered shall obey the same formula of definition as a combination having from the outset a signification, so that the old laws of reckoning shall still hold good and may still be applied to it.

A person who is competent to apply this principle rigorously and logically will arrive at combinations of numbers whose results are termed irrational or imaginary with the same necessity and facility as at the combinations above discussed, whose results are termed negative numbers and zero. To think of such combinations as *results* and to call the products reached also "numbers" is a misuse of language. It were better if we used the phrase *forms of numbers* for all numbers that are not the results of counting. But *usus tyrannus*!

It will now be my task to show how all numbers at which arithmetic ever has arrived or ever can arrive naturally flow from the simple application of the principle of no exception.

Owing to the commutative and associative laws for addition it is wholly indifferent for the result of a series of additive processes in what order the numbers to be summed are added. For example,

$$a+(b+c+d)+(e+f)=(a+b+c)+(d+e)+f. \tag{1}$$

The necessary consequence of this is that we may neglect the consideration of the order of the numbers and give heed only to what the quantities are that are to be summed, and, when they are equal, take note of only two things, namely, of what the quantity which is to be repeatedly summed is called and how often it occurs. We thus reach the notion of multiplication. To multiply a by b means to form the sum of b numbers each of which is called a. The number conceived summed is called the multiplicand, the number which indicates or counts how often the first is conceived summed is called the multiplier.

It appears hence, that the multiplier must be a result of counting, or a number in the original sense of the word, but that the multiplicand may be any number hitherto defined, that is, may also be zero or negative. It also follows from this definition that though the multiplicand may be a concrete number the multiplier cannot. Therefore, the commutative law of multiplication does not hold when the multiplicand is concrete. For, to take an example, though there is sense in requiring four trees to be summed three times, there is no sense in conceiving the number three summed "four trees times." When, however, multiplicand and multiplier are unnamed results of counting, (abstract numbers,) two fundamental laws hold in multiplication, exactly analogous to the fundamental laws of addition, namely, the law of commutation and the law of association. Thus,

$$\begin{aligned} a \textit{ times } b &= b \textit{ times } a, \\ \text{and, } a \textit{ times } (b \textit{ times } c) &= (a \textit{ times } b) \textit{ times } c \end{aligned}$$

The truth and correctness of these laws will be evident, if keeping to the definition of multiplication as an abbreviated addition of equal summands, we go back to the laws of addition. Owing to the commutative law it is unnecessary, for purposes of practical reckoning, to distinguish multiplicand and multiplier. Both have, therefore, a common name: *factor*. The result of the multiplication is called the product; the symbol of multiplication is a dot (·) or a cross (×), which is read "times." Joined with the fundamental formula above written are a group of subsidiary formulæ which give directions how a sum or difference is multiplied and how multiplication is performed with a sum or difference. I need not enter, however, into any discussion of these rules here.

As the combination of two numbers by a sign of multiplication has no significance according to our definition of multiplication, when the multiplier is zero or a negative number, it will be seen that we are again in a position where it is necessary to apply the above explained principle of no exception. We revert, therefore, to what we above established, that zero and negative numbers are symbols which have the form of differences, and lay down the rule that multiplications with zero and negative numbers shall be performed exactly as with real differences. Why, then, is minus one times minus one, for example, equal to plus one? For no other reason than that minus one can be multiplied with an ordinary difference, as, for example, 8 minus 5, by first multiplying by 8, then multiplying by 5, and subtracting the differences obtained, and because agreeably to the principle of no exception we must say that the multiplication must be performed according to exactly the same rule with a symbol which has the *form* of a difference whose minuend is less by one than its subtrahent.

As from addition two inverse operations, detraction and subtertraction, spring, so also from multiplication two inverse operations must proceed which differ from each other simply in the respect that in the one the multiplicand is sought and in the other the multiplier. As matters of computation, these two inverse operations coalesce in a single operation, namely, division, owing to the validity of the commutative law in multiplication. But in so far as they are different ideas, they must be distinguished. As most civilised languages distinguish the two inverse processes of multiplication in the case in which the multiplicand is a line, we will adopt for arithmetic a name which is used in this exception. Let us take this example,

$$4\,yards \times 3 = 12\,yards.$$

If twelve yards and four yards are given, and the multiplier 3 is sought, I ask, how many summands, each equal to four yards, give twelve yards, or, what is the same thing, how often I can lay a length of four yards on a length of twelve yards? But this is *measuring*. Secondly, if twelve yards and the number 3 are given, and the multiplicand four yards is sought, I ask what summand it is which taken three times gives twelve yards, or, what is the same thing, what part I shall obtain if I cut up twelve yards into three equal parts? But this is partition, or *parting*. If, therefore, the multiplier is sought we call the division *measuring*, and if the multiplicand is sought, we call it *parting*. In both cases the number which was originally the product is called the dividend, and the result the quotient. The number which originally was multiplicand is called the measure; the number which originally was multiplier is called the parter. The common name for measure and parter is divisor. The common symbol for both kinds of division is a colon, a horizontal stroke, or a combination of both. Its definitional formula reads,

$$(a \div b) \times b = a, \text{ or, } \frac{a}{b} \times b = a.$$

Accordingly, dividing a by b means, to find the number which multiplied by b gives a, or to find the number *with* which b must be multiplied to produce a. From this formula, together with the formulæ relative to multiplication, the well-known rules of division are derived, which I here pass over.

In the dividend of a quotient only such numbers can have a place which are the product of the divisor with some previously defined number. For example, if the divisor is 5 the dividend can only be 5, 10, 15, and so forth, and 0, -5, -10 and so forth. Accordingly, a stroke of division having underneath it 5 and above it a number different from the numbers just named is a combination of symbols having no meaning. For example, $\frac{3}{5}$ or $\frac{12}{5}$ are meaningless symbols. Now, conformably to the principle of no exception we must invest such symbols which have the form of a quotient without their dividend being the product of the divisor with any number yet defined, with a meaning such that we shall be able to reckon with such apparent quotients as with ordinary quotients. This is done by our agreeing always to put the product of such a quotient form with its divisor equal to its dividend. In this way we reach the definition of broken numbers or *fractions*, which by the application of the principle of no exception spring from division exactly as zero and negative numbers sprang from subtraction. The latter had their origin in the impossibility of the subtraction; the former have their origin in the impossibility of the division. Putting together now both these extensions of the domain of numbers, we arrive at *negative fractional numbers*.

We pass over the easily deduced rules of computation for fractions and shall only direct the reader's attention to the connexion which exists between fractional and non-fractional or, as we usually say, whole numbers. Since the number 12 lies between the numbers 10 and 15, or, what is the same thing, $10 < 12 < 15$, and since $10 : 5 = 2$, $15 : 5 = 3$, we say also that $12 : 5$ lies between 2 and 3, or that

$$2 < \frac{12}{5} < 3.$$

In itself, the notion of "less than" has significance only for results of counting. Consequently, it must first be stated what is meant when it is said that 2 is less than $\frac{12}{5}$. Plainly, nothing is meant by this except that 2 times 5 is less than 12. We thus see that every broken number can be so interpolated between two whole numbers differing from each other only by 1 that the one shall be smaller and the other greater, where smaller and greater have the meaning above given.

From the above definitions and the laws of commutation and association all possible rules of computation follow, which in virtue of the principle of no exception now hold indiscriminately for all numbers hitherto defined. It is a consequence of these rules, again, that the combination of two such numbers by means of any of the operations defined must in every case lead to a number which has been already defined, that is, to a positive or negative whole or fractional number, or to zero. The sole exception is the case where such a number is to be divided by zero. If the dividend also is zero, that is, if we have the combination $\frac{0}{0}$, the expression is one which stands for any number whatsoever, because any number whatsoever, no matter what it is, if multiplied by zero gives zero. But if the dividend is not zero but some other number a, be it what it will, we get a quotient form to which *no* number hitherto defined can be equated. But we discover that if we apply the ordinary arithmetical rules to $a \div 0$ all such forms may be equated to one another both when a is positive and also when a is negative. We may therefore invent two new signs for such quotient forms, namely $+\infty$ and $-\infty$. We find, further, that in transferring the notions greater and less to these symbols, $+\infty$ is greater than any positive number, however great, and $-\infty$ is smaller than any negative number, however small. We read these new signs, accordingly, "plus infinitely great" and "minus infinitely great."

But even here arithmetic has not reached its completion, although the combination of as many previously defined numbers as we please by as many previously defined operations as we please will still lead necessarily to some previously defined number. Every science must make every possible advance, and still one step in advance is possible in arithmetic. For in virtue of the laws of commutation and association, which also fortunately obtain in multiplication, just as we advance from addition to multiplication, so here again we may ascend from multiplication to *an operation of the third degree*. For, just as for $a+a+a+a$ we read $4 \cdot a$, so with the same reason we may introduce some more abbreviated designation for $a \cdot a \cdot a \cdot a$. The introduction of this new operation is in itself simply a matter of convenience and not an extension of the ideas of arithmetic. But if after having introduced this operation we repeatedly apply the monistic principle of arithmetic, the principle of no exception, we reach new means of computation which have led to undreamt of advances not only in the hands of mathematicians but also in the hands of natural scientists. The abbreviated designation mentioned, which, fructified by the principle of no exception, can render science such incalculable services, is simply that of writing for a product of b factors of which each is called a, a^b, which we read a to the b^{th} power. Here a new direct operation, that of *involution*, is defined, and from now on we are justified in distinguishing operations which are not inverses of others, as addition, multiplication, and involution, by *numbers of degree*. Addition is the direct operation of the first degree, multiplication that of the second degree, and involution that of the third degree. In the expression a^b the passive number a is called the *base*, the active number b the *exponent*, the result, the *power*.

But whilst in the direct operations of the first and second degree, the laws of commutation and association hold, here in involution, the operation of third degree, the two laws are inapplicable, and the result of their inapplicability is that operations of a still higher degree than the third form no possible advancement of pure arithmetic. The product of b factors a is not equal to the product of a factors b; that is, the law of commutation does not hold. The only two different integers for which a to the b^{th} power is equal to b to the a^{th} power are 2 and 4, for 2 to the 4^{th} power is 16, and 4 to the second power also is 16. So, too, the law of association as a general rule does not hold. For it is hardly the same thing whether we take the $(b^c)^{th}$ power of a or the c^{th} power of ab.

From the definition of involution follow the usual rules for reckoning with powers, of which we shall only mention one, namely, that the $(b-c)^{th}$ power of a is equal to the result of the division of a to the b^{th} power by a to the c^{th} power. If we put here c equal to b, we are obliged, by the principle of no exception, to put a to the 0^{th} power equal to 1; a new result not contained in the original notion of involution, for that implied necessarily that the exponent should be a result of counting. Again, if we make b smaller than c we obtain a *negative exponent*, which we should not know how to dispose of if we did not follow our monistic law of arithmetic. According to the latter, a to the $(b-c)^{th}$ power must still remain equal to a^b divided by a^c even when b is smaller than c. Whence follows that a to the minus d^{th} power is equal to 1 divided by a to the d^{th} power, or to take specific numbers, that 3 to the minus 2^{nd} power is equal to $\frac{1}{9}$.

At this point, perhaps, the reader will inquire what a raised to a fractional power is. But this can be explained only when we have discussed the inverse processes of involution, to which we now pass.

If $a^b = c$, we may ask two questions: first, what the base is which raised to the b^{th} power gives c; the second, what the exponent of the power is to which a must be raised to produce c. In the first case we seek the base, and term the operation which yields this result *evolution*; in the second case we seek the exponent and call the operation which yields this exponent, the *finding of the logarithm*. In the first case, we write $\sqrt[b]{c} = a$ (which we read, the b^{th} root of c is equal to a), and call c the *radicand*, b the *exponent of the root*, and a the root. In the second case, we write $\log_a c = b$ (which we read, the logarithm of c to the base a is equal to b), and call c the *logarithmand* or *number*, a the *base of the logarithm*, and b the *logarithm*.

While, owing to the validity of the law of commutation in addition and multiplication, the two inverse processes of those operations are identical so far as computation is concerned, here in the case of involution the two inverse operations are in this regard essentially different, for in this case the law of commutation does not hold.

From the definitional formulæ for evolution and the finding of logarithms, namely,

$$(\sqrt[b]{c})^b = c, \text{ and } (a)^{\log_a c} = c$$

follow, by the application of the laws of involution, the rules for computation with roots and logarithms. These rules we pass over here, only remarking, first, that for the present $\sqrt[b]{c}$ has meaning only when c is the b^{th} power of some number already defined; and, secondly, that for the present also $\log_a c$ has meaning only when c can be produced by raising the number a to some power which is a number already defined. In the phrase "has only meaning for the present" is contained a possibility of new extensions of the domain of number. But before we pass to those extensions we shall first make use of the idea of evolution just defined to extend the notion of power also to cases in which the exponent is a fractional number.

According to the original definition of involution, *ab* was meaningless except where *b* was a result of counting. But afterwards, even powers which had for their exponents zero or a negative integer could be invested with meaning. Now we have to consider the arithmetical combination "a raised to the fractional power $\frac{p}{q}$." The principle of no exception compels us to give to the arithmetical combination "a to the $\frac{p}{q}^{th}$ power" a significance such that all the rules of computation will hold with respect to it. Now, one rule that holds is, that the m^{th} power of the n^{th} power of a is equal to the $(m \times n)^{th}$ power of a. Consequently, the q^{th} power of a raised to the $\frac{p}{q}^{th}$ power must be equal to a raised to a power whose exponent is equal to $\frac{p}{q}$ times q. But the last-mentioned product gives, according to the definition of division, the number p. Consequently the symbol a to the $\frac{p}{q}^{th}$ power is so constituted that its q^{th} power is equal to a to the p^{th} power; i. e., it is equal to the q^{th} root of a^p. Similarly, we find that the symbol "a to the minus $\frac{p}{q}^{th}$ power" must be put equal to 1 divided by the q^{th} root of a to the p^{th} power, if we are to reckon with this symbol as we do with real powers. Again, just as a to the b^{th} power is invested with meaning when b is a fractional number, so some meaning harmonious with the principle of no exception must be imparted to the b^{th} root of c where b is a positive or negative fractional number. For example, the three-fourthsth root of 8 is equal to 8 to the $\frac{4}{3}$ power, that is, to the cube root of 8 to the 4^{th} power, or 16.

The principle underlying arithmetic now also compels us to give to the symbol the "b^{th} root of c" a meaning when c is not the b^{th} power of any number yet defined. First, let c be any *positive* integer or fraction. Then always to be able to reckon with the b^{th} root of c in the same way that we do with extractible roots, we must agree always to put the b^{th} power of the b^{th} root of c equal to c—for example,$(\sqrt[2]{3})^2$ always exactly equal to 3. A careful inspection of the new symbols, which we will also call numbers, shows, that though no one of them is exactly equal to a number hitherto defined, yet by a certain extension of the notions greater and less, two numbers of the character of numbers already defined may be found for each such new number, such that the new number is greater than the one and less than the other of the two, and that further, these two numbers may be made to differ from each other by as small a quantity as we please. For example,

$$\left(\frac{7}{5}\right)^3 = \frac{343}{125} = 2\frac{93}{125} < 3 < 3\frac{3}{8} = \frac{27}{8} = \left(\frac{3}{2}\right)^3$$

The number 3, as we see, is here included between two limits which are the third powers of two numbers $\frac{7}{5}$ and $\frac{3}{2}$ whose difference is $\frac{1}{10}$. We could also have arranged it so that the difference should be equal to $\frac{1}{100}$, or to any specified number, however small. Now, instead of putting the symbol "less than" between $\left(\frac{7}{5}\right)^3$ and 3, and between 3 and $\left(\frac{3}{2}\right)^3$, let us put it between their third roots; for example, let us say:

$$\frac{7}{5} < \sqrt[3]{3} < \frac{3}{2}, \text{ meaning by this that} \left(\frac{7}{5}\right)^3 < 3 < \left(\frac{3}{2}\right)^3$$

In this sense we may say that the new numbers always lie *between* two old numbers whose difference may be made as small as we please. Numbers possessing this property are called *irrational* numbers, in contradistinction to the numbers hitherto defined, which are termed *rational*. The considerations which before led us to negative rational numbers, now also lead us to negative irrational numbers. The repeated application of addition and multiplication as of their inverse processes to irrational numbers, (numbers which though not exactly equal to previously defined rational numbers may yet be brought as near to them as we please,) again simply leads to numbers of the same class.

A totally new domain of numbers is reached, however, when we attempt to impart meaning to *the square roots of negative numbers*. The square root of minus 9 is neither equal to plus 3 nor to minus 3, since each multiplied by itself gives plus 9, nor is it equal to any other number hitherto defined. Accordingly, the square root of minus 9 is a new number-form, to which, harmoniously with the principle of no exception, we may give the definition that $(\sqrt[2]{-9})^2$ shall always be put equal to minus 9.[3] Keeping to this definition we see at once that $\sqrt{-a}$, where a is any positive rational or irrational number, is a symbol which can be put equal to the product of $\sqrt{+a}$ by $\sqrt{-1}$. In extending to these new numbers the rights of arithmetical citizenship, in calling them also "numbers," and so shaping their definition that we can reckon with them by the same rules as with already defined numbers, we obtain a fourth extension of the domain of numbers which has become of the greatest importance for the progress of all branches of mathematics. The newly defined numbers are called *imaginary*, in contradistinction to all heretofore defined, which are called real . Since all imaginary numbers can be represented as products of real numbers with the square root of minus one, it is convenient to introduce for this one imaginary number some concise symbol. This symbol is the first letter of the word imaginary, namely, i; so that we can always put for such an expression as $\sqrt{-9}$, $3 \cdot i$.

If we combine real and imaginary numbers by operations of the first and second degree, always supposing that we follow in our reckoning with imaginary numbers the same rules that we do in reckoning with real numbers, we always arrive again at real or imaginary numbers, excepting when we join together a real and an imaginary number by addition or its inverse operations. In

[3] Henceforward we shall use the simpler sign $\sqrt{}$ for $\sqrt[2]{}$.

this case *we reach the symbol* $a+i\cdot b$, where a and b stand for real numbers. Agreeably to the principle of no exception we are permitted to reckon with $a+ib$ according to the same rules of computation as with symbols previously defined, if for the second power of i we always substitute minus 1.

In the numerical combination $a+ib$, which we also call number, we have found the most general numerical form to which the laws of arithmetic can lead, even though we wished to extend the limits of arithmetic still further. Of course, we must represent to ourselves here by a and b either zero or positive or negative rational or irrational numbers. If b is zero, $a+ib$ represents all real numbers; if a is zero, it stands for all purely imaginary numbers. This general number $a+ib$ is called a *complex number*, so that the complex number includes in itself as special cases all numbers heretofore defined. By the introduction of irrational, purely imaginary, and the still more general complex numbers, all combinations become invested with meaning which the operations of the third degree can produce. For example, the fifth root of 5 is an irrational number, the logarithm of 2 to the base 10 is an irrational number. The logarithm of minus 1 to the base 2 is a purely imaginary number; the fourth root of minus 1 is a complex number. Indeed, we may recognise, proceeding still further, *that every combination of two complex numbers, by means of any of the operations of the first, second, or third degree will lead in turn to a complex number*, that is to say, never furnishes occasion, by application of the principle of no exception, for inventing new forms of numbers.

A certain limit is thus reached in the construction of arithmetic. But such a limit was also twice previously reached. After the investigation of addition and its inverse operations, we reached no other numbers except zero and positive and negative whole numbers, and every combination of such numbers by operations of the first degree led to no new numbers. After the investigation of multiplication and its inverse operations, the positive or negative fractional numbers and "infinitely great" were added, and again we could say that the combination of two already defined numbers by operations of the first and second degree in turn also always led to numbers already defined. Now we have reached a point at which we can say that the combination of two complex numbers by all operations of the first, second, and third degree must again always lead to complex numbers; only that now such a combination does not necessarily always lead to a single number, but may lead to many regularly arranged numbers. For example, the combination "logarithm of minus one to a positive base" furnishes a countless number of results which form an arithmetical series of purely imaginary numbers. *Still, in no case now do we arrive at new classes of numbers*. But just as before the ascent from multiplication to involution brought in its train the definition of new numbers, so it is also possible that *some new operation springing out of involution as involution sprang from* *multiplication might furnish the germ of other new numbers which are not reducible to $a+ib$.* As a matter of fact, mathematicians have asked themselves this question and investigated the direct operation of the fourth degree, together with its inverse processes. The result of their investigations was, that an operation which springs from involution as involution sprang from multiplication is incapable of performing any real mathematical service; the reason of which is, that in involution the laws of commutation and association do not hold. It also further appeared that the operations of the fourth degree could not give rise to new numbers. No more so can operations of still higher degrees. With respect to quaternions, which many might be disposed to regard as new numbers, it will be evident that though quaternions are valuable means of investigation in geometry and mechanics they are not numbers of arithmetic, because the rules of arithmetic are not unconditionally applicable to them.

The building up of arithmetic is thus completed. The extensions of the domain of number are ended. It only remains to be asked why the science of arithmetic appears in its structure so logical, natural, and unarbitrary; why zero, negative, and fractional numbers appear as much derived and as little original as irrational, imaginary, and complex numbers? We answer, wholly and alone in virtue of the logical application of the monistic principle of arithmetic, the principle of no exception.

The Fourth Dimension. Mathematical and Spiritualistic

I. Introductory

THE TENDENCY TO GENERALISE long ago led mathematicians to extend the notion of three-dimensional space, which is the space of sensible representation, and to define aggregates of points, or spaces, of more than three dimensions, with the view of employing these definitions as useful means of investigation. They had no idea of requiring people to imagine four-dimensional things and worlds, and they were even still less remote from requiring them to believe in the real existence of a four-dimensioned space. In the hands of mathematicians this extension of the notion of space was a mere means devised for the discovery and expression, by shorter and more convenient ways, of truths applicable to common geometry and to algebra operating with more than three unknown quantities. At this stage, however, the spiritualists came in, and coolly took possession of this private property of the mathematicians. They were in great perplexity as to where they should put the spirits of the dead. To give them a place in the world accessible to our senses was not exactly practicable. They were compelled, therefore, to look around after some *terra incognita*, which should oppose to the spirit of research inborn in humanity an insuperable barrier. The abiding-place of the spirits had perforce to be inaccessible to the senses and full of mystery to the mind. This property the four-dimensioned space of the mathematicians possessed. With an intellectual perversity which science has no idea of, these spiritualists boldly asserted, first, that the whole world was situated in a four-dimensioned space as a plane might be situated in the space familiar to us, secondly, that the spirits of the dead lived in such a four-dimensioned space, thirdly, that these spirits could accordingly act upon the world and, consequently, upon the human beings resident in it, exactly as we three-dimensioned creatures can produce effects upon things that are two-dimensioned; for example, such effects as that produced when we shatter a lamina of ice, and so influence some possibly existing two-dimensioned *ice-*

world. Since spiritualism, under the leadership of a Leipsic Professor, Zöllner, thus proclaimed the existence of a four-dimensioned space, this notion, which the mathematicians are thoroughly master of,—for in all their operations with it, though they have forsaken the path of actual representability, they have never left that of the truth,—this notion has also passed into the heads of lay persons who have used it as a catchword, ordinarily without having any clear idea of what they or any one else mean by it. To clear up such ideas and to correct the wrong impressions of cultured people who have not a technical mathematical training, is the purpose of the following pages. A similar elucidation was aimed at in the tracts which Schlegel (Riemann, Berlin, 1888) and Cranz (Virchow-Holtzendorff's Sammlung, Nos. 112 and 113) have published on the so-called fourth dimension. Both treatises possess indubitable merits, but their methods of presentation are in many respects too concise to give to lay minds a profound comprehension of the subject. The author, accordingly, has been able to add to the reflections which these excellent treatises offer, a great deal that appears to him necessary for a thorough explanation in the minds of non-mathematicians of the notion of the fourth dimension.

II The Concept of Dimension

Many text-books of stereometry begin with the words: "Every body has three dimensions, length, breadth, and thickness." If we should ask the author of a book of this description to tell us the length, breadth, and thickness of an apple, of a sponge, or of a cloud of tobacco smoke, he would be somewhat perplexed and would probably say, that the definition in question referred to something different. A cubical box, or some similar structure, whose angles are all right angles and whose bounding surfaces are consequently all rectangles is the only body of which it can at all be unmistakably asserted that there are three principal directions distinguishable in it, of which any one can be called the length, any other the breadth, and any third the thickness. We thus see that the notions of length, breadth, and thickness are not sufficiently clear and universal to enable us to derive from them any idea of what is meant when it is said that every body possesses three dimensions, or that the space of the world is three-dimensional.

This distinction may be made sharper and more evident by the following considerations: We have, let us suppose, a straight line on which a point is situated, and the problem is proposed to determine the position of the point on the line in an unequivocal manner. The simplest way to solve this is, to state how far the point is removed in the one or the other direction from some given fixed point; just as in a thermometer the position of the surface of the mercury is given by a statement of its distance in the direction of cold or heat from a predetermined fixed point—the point of freezing water. To state, therefore, the position of a point on a straight line, the sole datum necessary is a single number, if beforehand we have fixed upon some standard line, like the centimetre, and some definite point to which we give the value zero, and have also previously decided in what direction from the zero-point, points must be situated whose position is expressed by positive numbers, and also in what direction those must lie whose position is expressed by negative numbers. This last-mentioned fact, that a *single* number is sufficient to determine the place of a point in a straight line, is the real reason why we attribute to the straight line or to any part of it a single dimension.

More generally, we call every totality or system, of infinitely numerous things, *one*-dimensional, in which *one* number is all that is requisite to determine and mark out any particular one of these things from among the entire totality. Thus, time is one-dimensional. We, as inhabitants of the earth, have naturally chosen as our unit of time, the period of the rotation of the earth about its axis, namely, the day, or a definite portion of a day. The zero-point of time is regarded in Christian countries as the year of the birth of Christ, and the positive direction of time is the time *subsequent* to the birth of Christ. These data fixed, all that is necessary to establish and distinguish any definite point of time amid the infinite totality of all the points of time, *is a single number*. Of course this number need not be a whole number, but may be made up of the sum of a whole number and a fraction in whose numerator and denominator we may have numbers as great as we please. We may, therefore, also say that the totality of all conceivable numerical magnitudes, or of only such as are greater than one definite number and smaller than some other definite number, is one-dimensional.

We shall add here a few additional examples of one-dimensional magnitudes presented by geometry. First, the circumference of a circle is a one-dimensional magnitude, as is every curved line, whether it returns into itself or not. Further, the totality of all equilateral triangles which stand on the same base is one-dimensional, or the totality of all circles that can be described through two fixed points. Also, the totality of all conceivable cubes will be seen to be one-dimensional, provided they are distinguished, not with respect to position, but with respect to magnitude.

In conformity with the fundamental ideas by which we define the notion of a one-dimensional manifoldness, it will be seen that the attribute *two*-dimensional must be applied to all totalities of things in which *two* numbers are necessary (and sufficient) to distinguish any determinate individual thing amid the totality. The simplest twodimensioned complex which we know of is the plane. To determine accurately the position of a point in a plane, the simplest way is to take two axes at right angles to each other, that is, fixed straight lines, and then to specify the distances by which the point in question is removed from each of these axes.

This method of determining the position of a point in a plane suggested to the celebrated philosopher and mathematician Descartes the fundamental idea of analytical geometry, a branch of mathematics in which by the simple artifice of ascribing to every point in a plane two numerical values, determined by its distances from the two axes above referred to, planimetrical considerations are transformed into algebraical. So, too, all kinds of curves that graphically represent the dependence of things on time, make use of the fact that the totality of the points in a plane is two-dimensional. For example, to represent in a graphical form the increase in the population of a city, we take a horizontal axis to represent the time, and a perpendicular one to represent the numbers which are the measures of the population. Any two lines, then, whose lengths practical considerations determine, are taken as the unit of time, which we may say is a year, and as the unit of population, which we will say is one thousand. Some definite year, say 1850, is fixed upon as the zero point. Then, from all the equally distant points on the horizontal axis, which points stand for the years, we proceed in directions parallel to the other axis, that is, in the perpendicular direction, just so much upwards as the numbers which stand for the population of that year require. The terminal points so reached, or the curve which runs through these terminal points, will then present a graphic picture of the rates of increase of the population of the town in the different years. The rectangular axes of Descartes are employed in a similar way for the construction of barometer curves, which specify for the different localities of a country the amount of variation of the atmospheric pressure during any period of time. Immediately next to the plane the surface of the earth will be recognised as a two-dimensional aggregate of points. In this case geographical latitude and longitude supply the two numbers that are requisite accurately to determine the position of a point. Also, the totality of all the possible straight lines that can be drawn through any point in space is two-dimensional, as we shall best understand if we picture to ourselves a plane which is cut in a point by each of these straight lines and then remember that by such a construction every point on the plane will belong to some one line and, *vice versa*, a line to every point, whence it follows that the totality of all the straight lines, or, as we may call them, rays, which pass through the point assigned are of the same dimensions as the totality of the points of the imagined plane.

The question might be asked, In what way and to what extent in this case is the specification of *two* numbers requisite and sufficient to determine amid all the rays which pass through the specified point a definite individual ray? To get a clear idea of the problem here involved, let us imagine the ray produced far into the heavens where some quite definite point will correspond to it. Now, the position of a point in the heavens depends, as does the position of a point on all spherical surfaces, on two numbers. In the heavens these two numbers are ordinarily supplied by the two angles called altitude, or the distance above the plane of the horizon, and azimuth, or the angular distance between the circle on which the altitude is measured and the meridian of the observer. It will be seen thus that the totality of all the luminous rays that an eye, conceived as a point, can receive from the outer world is two-dimensional, and also that a luminous point emits a two-dimensional group of luminous rays. It will also be observed, in connexion with this example, that the two-dimensional totality of all the rays that can be drawn through a point in space is something different from the totality of the rays that pass through a point but are required to lie in a given plane. Such a group of objects as the last-named one is a one-dimensional totality.

Now that we have sufficiently discussed the attributes that are characteristic of one and two-dimensional aggregates, we may, without any further investigation of the subject, propose the following definition, that, generally, *an n-dimensional totality of infinitely numerous things is such that the specification of n numbers is necessary and sufficient to indicate definitely any individual amid all the infinitely numerous individuals of that totality*.

Accordingly, the point-aggregate made up of the world-space which we inhabit, is a three-dimensional totality. To get our bearings in this space and to define any determinate point in it, we have simply to lay through any point which we take as our zero-point three axes at right angles to each other, one running from right to left, one backwards and forwards, and one upwards and downwards. We then join each two of these axes by a plane and are enabled thus to specify the position of every point in space by the three perpendicular distances by which the point in question is removed in a positive or negative sense from these three planes. It is customary to denote the numbers which are the measures of these three distances by x, y, and z, the positive x, positive y, and positive z ordinarily being reckoned in the right hand, the hitherward, and the upward directions from the origin. If now, with direct reference to this fundamental axial system, any particular specification of x, y, and z be made, there will, by such an operation, be cut out and isolated from the three-dimensional manifoldness of all the points of space a totality of less dimensions. If, for example, z is equal to seven units or measures, this is equivalent to a statement that only the two-dimensional totality of the points is meant, which constitute the plane that can be laid at right angles to the upward-passing z-axis at a distance of seven measures from the zero-point. Consequently, every imaginable equation between x, y, and z isolates and defines a two-dimensional aggregate of points. If two different equations obtain between x, y, and z, two such two-dimensional totalities will be isolated from among all the points of space. But as these last must have some one-dimensional totality in common, we may say that the co-existence of two equations between x, y, and z defines a one-dimensional totality of points, that is to say a straight line, a line curved in a plane, or even, perhaps, one curved in space. It is evident from this that the introduction of the three axes of reference forms a bridge between the theory of space and the theory of equations involving three variable quantities, x, y, z. The reason that the theory of space cannot thus be brought into connection with algebra in general, that is, with the theory of indefinitely numerous equations, but only with the algebra of three quantities, x, y, z, is simply to be sought in the fact that space, as we picture it, can have only three dimensions.

We have now only to supply a few additional examples of ndimensional totalities. All particles of air are four-dimensional in magnitude when in addition to their position in space we also consider the variable densities which they assume, as they are expressed by the different heights of the barometer in the different parts of the atmosphere. Similarly, all conceivable spheres in space are fourdimensional magnitudes, for their centres form a three-dimensional point-aggregate, and around each centre there may be additionally conceived a one-dimensional totality of spheres, the radii of which can be expressed by every numerical magnitude from zero to infinity. Further, if we imagine a measuring-stick of invariable length to assume every conceivable position in space, the positions so obtained will constitute a five-dimensional aggregate. For, in the first place, one of the extremities of the measuring stick may be conceived to assume a position at every point of space, and this determines for one extremity alone of the stick a three-dimensional totality of positions; and secondly, as we have seen above, there proceeds from every such position of this extremity a two-dimensional totality of directions, and by conceiving the measuring-stick to be placed lengthwise in every one of these directions we shall obtain all the conceivable positions which the second extremity can assume, and consequently, the dimensions must be 3 plus 2 or 5. Finally, to find out how many dimensions the totality of all the possible positions of a square, invariable in magnitude, possesses, we first give one of its corners all conceivable positions in space, and we thus obtain three dimensions. One definite point in space now being fixed for the position of one corner of the square, we imagine drawn through this point all possible lines, and on each we lay off the length of the side of the square and thus obtain two additional dimensions. Through the point obtained for the position of the second corner of the square we must now conceive all the possible directions drawn that are perpendicular to the line thus fixed, and we must lay off once more on each of these directions the side of the square. By this last determination the dimensions are only increased by one, for only one one-dimensional totality of perpendicular directions is possible to one straight line in one of its points. Three corners of the square are now fixed and therewith the position of the fourth also is uniquely determined. Accordingly, the totality of all equal squares which differ from one another only by their position in space, constitutes a manifoldness of six dimensions.

III. The Introduction of the Notion of Four-Dimensional Point-Aggregates, Permissible.

In the preceding section it was shown that we can conceive not only of manifoldnesses of one, two, and three dimensions, but also of manifoldnesses of *any* number of dimensions. But it was at the same time indicated that our world-space, that is, the totality of all conceivable *points* that differ only in respect of position, cannot in agreement with our notions of things possess more than three dimensions. But the question now arises, whether, if the progress of science tends in such a direction, it is permissible to extend the notion of space by the introduction of point-aggregates of more than three dimensions, and to engage in the study of the properties of such creations, although we know that notwithstanding the fact that we may conceptually establish and explore such aggregates of points, yet we cannot picture to ourselves these creations as we do the spatial magnitudes which surround us, that is, the regular three-dimensional aggregates of points.

To show the reader clearly that this question must be answered in the affirmative, that the extension of our notion of space is permissible, although it leads to things which we cannot perceive by our senses, I may call the reader's attention to the fact that in arithmetic we are accustomed from our youth upwards to extensions of ideas, which, accurately viewed, as little admit of graphic conception as a four-dimensional space, that is, a point-aggregate of four dimensions. By his senses man first reaches only the idea of whole numbers—the results of counting. The observation of primitive peoples[4] and of children clearly proves that the essential decisive factors of counting are these three: First, we abstract, in the counting of things, completely from the individual and characteristic attributes of these things, that is, we consider them as homogeneous. Second, we associate individually with the things which we count other homogeneous things. These other things are even now, among uncivilised peoples, the ten fingers of the two hands. They may, however, be simple strokes, or, as in the case of dice and dominoes, black points on a white background. Third, we substitute for the result of this association some concise

[4]See the essay Notion and Definition of Number in this collection.

symbol or word; for example, the Romans substituted for three things counted, three strokes placed side by side, namely: III; but for greater numbers of things they employed abbreviated signs. The Aztecs, the original inhabitants of Mexico, had time enough, it seems, to express all the numbers up to nineteen by equal circles placed side by side. They had abbreviated signs only for the numbers 20, 400, 8000, and so forth. In speaking, some one same sound might be associated with the things counted; but this method of counting is nowadays employed only by clocks: the languages of men since prehistoric times have fashioned concise words for the results of the association in question. From the notion of number, thus fixed as the result of counting, man reached the notion of the addition of two numbers, and thence the notion that is the inverse of the last process, the notion of subtraction. But at this point it clearly appears that not every problem which may be propounded is soluble; for there is no number which can express the result of the subtraction of a number from one which is equally large or from one which is smaller than itself. The primary school pupil who says that 8 from 5 "won't go" is perfectly right from his point of view. For there really does not exist any result of counting which added to eight will give five.

If humanity had abided by this point of view and had rested content with the opinion that the problem "5 minus 8" is not solvable, the science of arithmetic would never have received its full development, and humanity would not have advanced as far in civilisation as it has. Fortunately, men said to themselves at this crisis: "If 5 minus 8 won't go, we'll *make it* go; if 5 minus 8 does not possess an intelligible meaning, we will simply give it one." As a fact, things which have not a meaning always afford men a pleasing opportunity of investing them with one. The question is, then, what significance is the problem "5 minus 8" to be invested with?

The most natural and therefore the most advantageous solution undoubtedly is to abide by the original notion of subtraction as the inverse of addition, and to make the significance of 5 minus 8 such, that for 5 minus 8 plus 8 we shall get our original minuend 5. By such a method all the rules of computation which apply to real differences will also hold good for unreal differences, such as 5 minus 8. But it then clearly appears that all forms expressive of differences in which the numbers that stand before the minus symbol are less by an equal amount than those which follow it may be regarded as equal; so that the simplest course seems to be to introduce as the common characteristic of all equal differential forms of this description a common sign, which will indicate at the same time the difference of the two numbers thus associated. Thus it came about that for 5 minus 8, as well as for every differential form which can be regarded as equal thereto, the sign "-3" was introduced. But in calling differential forms of this description numbers, the notion of number was extended and a new domain was opened up, namely the domain of negative numbers.

In the further development of the science of arithmetic, through the operation of division viewed as the inverse of multiplication, a second extension of the idea of number was reached, namely, the notion of fractional numbers as the outcome of divisions that had led to numbers hitherto undefined. We find, thus, that the science of arithmetic throughout its whole development has strictly adhered to the principle of conformity and consistency and has invested every association of two numbers, which before had no significance, by the introduction of new numbers, with a real significance, such that similar operations in conformity with exactly the same rules could be performed with the new numbers, viewed as the results of this association, as with the numbers which were before known and perfectly defined. Thus the science proceeded further on its way and reached the notions of irrational, imaginary, and complex numbers.

The point in all this, which the reader must carefully note, is, that all the numbers of arithmetic, with the exception of the positive whole numbers, are artificial products of human thought, invented to make the language of arithmetic more flexible, and to accelerate the progress of science. All these numbers lack the attributes of representability.

No man in the world can picture to himself "minus three trees." It is possible, of course, to know that when three trees of a garden have been cut down and carried away, three are missing, and by substituting for "missing" the inverse notion of "added," we may say, perhaps, that "minus three trees" are added. But this is quite different from the feat of imagining a negative number of trees. We can only picture to ourselves a number of trees that results from actual counting, that is, a positive whole number. Yet, notwithstanding all this, people had not the slightest hesitation in extending the notion of number. Exactly so must it be permitted us in geometry to extend the notion of space, even though such an extension can only be mentally defined and can never be brought within the range of human powers of representation.

In mathematics, in fact, the extension of any notion is admissible, provided such extension does not lead to contradictions with itself or with results which are well established. Whether such extensions are necessary, justifiable, or important for the advancement of science is a different question. It must be admitted, therefore, that the mathematician is justified in the extension of the notion of space as a point-aggregate of three dimensions, and in the introduction of space or point-aggregates of more than three dimensions, and in the employment of them as means of research. Other sciences also operate with things which they do not know exist, and which, though they are sufficiently defined, cannot be perceived by our senses. For example, the physicist employs the ether as a means of investigation, though he can have no sensory knowledge of it. The ether is nothing more than a means which enables us to comprehend mechanically the effects known as action at a distance and to bring them within the range of a common point of view. Without the assumption of a material which penetrates everything, and by means of whose undulations impulses are transmitted to the remotest parts of space, the phenomena of light, of heat, of gravitation, and of electricity would be a jumble of isolated and unconnected mysteries. The assumption of an ether, however, comprises in a systematic scheme all these isolated events, facilitates our mental control of the phenomena of nature, and enables us to produce these phenomena at will. But it must not be forgotten in such reflexions that the ether itself is even a greater problem for man, and that the ether-hypothesis does not solve the difficulties of phenomena, but only puts them in a unitary conceptual shape. Notwithstanding all this, physicists have never had the least hesitation in employing the ether as a means of investigation. And as little do reasons exist why the mathematicians should hesitate to investigate the properties of a four-dimensioned point-aggregate, with the view of acquiring thus a convenient means of research.

IV. The Introduction of the Idea of Four-Dimensioned Point-Aggregates of Service to Research.

From the concession that the mathematician has the right to define and investigate the properties of point-aggregates of more than three dimensions, it does not necessarily follow that the introduction of an idea of this description is of value to science. Thus, for example, in arithmetic, the introduction of operations which spring from involution, as involution and its two inverse operations proceed from multiplication, is undoubtedly permitted. Just as for "*a* times *a* times *a*" we write the abbreviated symbol "a^3," (which we read, *a* to the third power,) and investigate in detail the operation of involution thus defined, so we might also introduce some shorthand symbol for "*a* to the a^{th} power to the a^{th} power" and thus reach an operation of the fourth degree, which would regard a as a passive number and the number 3, or any higher number, as the active number, that is, as the number which indicates how often *a* is taken as the base of a power whose exponent may be *a*, or "*a* to the a^{th} , or "*a* to the a^{th} to the a^{th} power."

But the introduction of such an operation of the fourth degree has proved itself to be of no especial value to mathematics. And the reason is that in the operation of involution the law of commutation does not hold good. In addition, the numbers to be added may be interchanged and the introduction of multiplication is therefore of great value. So, also, in multiplication the numbers which are combined, that is, the factors, may be changed about in any way, and thus the introduction of involution is of value. But in involution the base and the exponent cannot be interchanged, and consequently the introduction of any higher operation is almost valueless.

But with the introduction of the idea of point-aggregates of multiple dimensions the case is wholly different. The innovation in question has proved itself to be not only of great importance to research, but the progress of science has irresistibly forced investigators to the introduction of this idea, as we shall now set forth in detail.

In the first place, algebra, especially the algebraical theory of systems of equations, derives much advantage from the notion of multidimensioned spaces. If we have only three unknown quantities, x, y, z, the algebraical questions which arise from the possible problems of this class admit, as we have above seen, of geometrical representation to the eye. Owing to this possibility of geometrical representation, some certain simple geometrical ideas like "moving," "lying in," "intersecting," and so forth, may be translated into algebraical events. Now, no reason exists why algebra should stop at three variable quantities; it must in fact take into consideration any number of variable quantities.

For purposes of brevity and greater evidentness, therefore, it is quite natural to employ geometrical forms of speech in the consideration of more than three variables. But when we do this, we assume, perhaps without really intending to do so, the idea of a space of more than three dimensions. If we have four variable quantities, x, y, z, u, we arrive, by conceiving attributed to each of these four quantities every possible numerical magnitude, at a four-dimensioned manifoldness of numerical quantities, which we may just as well regard as a four-dimensioned aggregate of points. Two equations which exist on this supposition between x, y, z, and u, define two three-dimensioned aggregates of points, which intersect, as we may briefly say, in a two-dimensioned aggregate of points, that is, in a surface; and so on. In a somewhat different manner the determination of the contents of a square or a cube by the involution of a number which stands for the length of its sides, leads to the notion of four-dimensioned structures, and, consequently, to the notion of a four-dimensioned point-space. When we note that a^2 stands for the contents of a square, and a^3 for the content of a cube, we naturally inquire after the contents of a structure which is produced from the cube as the cube is produced from the square and which also will have the contents a^4. We cannot, it is true, clearly picture to ourselves a structure of this description, but we can, nevertheless, establish its properties with mathematical exactness.[5] It is bounded by 8 cubes just as the cube is bounded by 6 squares; it has 16 corners, 24 squares, and 32 edges, so that from every corner 4 edges, 6 squares, and 4 cubes proceed, and from every edge 3 squares and 3 cubes.

Yet despite the great service to algebra of this idea of multidimensioned space, it must be conceded that the conception although convenient is yet not indispensable. It is true, algebra is in need of the idea of multiple dimensions,

[5]Victor Schlegel, indeed, has made models of the three-dimensional nets of all the six structures which correspond in four-dimensioned space to the five regular bodies of our space, in an analogous manner to that by which we draw in a plane the nets of five regular bodies. Schlegel's models are made by Brill of Darmstadt.

but it is not so absolutely in need of the idea of *point*-aggregates of multiple dimensions.

This notion is, however, necessary and serviceable for a profound comprehension of geometry. The system of geometrical knowledge which Euclid of Alexandria created about three hundred years before Christ, supplied during a period of more than two thousand years a brilliant example of a body of conclusions and truths which were mutually consistent and logical. Up to the present century the idea of elementary geometry was indissolubly bound up with the name of Euclid, so that in England where people adhered longest to the rigid deductive system of the Grecian mathematician, the task of “learning geometry” and “reading Euclid” were until a few years ago identical. Every proposition of this Euclidean system rests on other propositions, as one building-stone in a house rests upon another. Only the very lowest stones, the foundations, were without supports. These are the axioms or fundamental propositions, truths on which all other truths are, directly or indirectly, founded, but which themselves are assumed without demonstration as self-evident.

But the spirit of mathematical research grew in time more and more critical, and finally asked, whether these axioms might not possibly admit of demonstration. Especially was a rigid proof sought for the eleventh[6] axiom of Euclid, which treats of parallels.

After centuries of fruitless attempts to prove Euclid’s eleventh axiom, Gauss, and with him Bolyai and Lobachévski, Riemann, and Helmholtz, finally stated the decisive reasons why any attempt to prove the axiom of the parallels must necessarily be futile. These reasons consist of the fact that though this axiom holds good enough in the world-space such as we do and can conceive it, yet three-dimensioned spaces are ideally conceivable though not capable of mental representation, where the axiom does not hold good. The axiom was thus shown to be a mere fact of *observation*, and from that time on there could no longer be any thought of a deductive demonstration of it. In view of the intimate connection, which both in an historical and epistemological point of view exists between the extension of the concept of space and the critical examination of the axioms of Euclid, we must enter at somewhat greater length into the discussion of the last mentioned propositions.

Of the axioms which Euclid lays at the foundation of his geometry, only the following three are really geometrical axioms:

Eighth axiom: Magnitudes which coincide with one another are equal to one another.

[6] Also called the twelfth axiom, also the fifth postulate.–Tr.

Eleventh axiom: If a straight line meet two straight lines so as to make the two interior angles on the same side of it taken together less than two right angles, these straight lines, being continually produced, shall at length meet on that side on which are the angles which are less than two right angles.

Twelfth axiom: Two straight lines cannot inclose a [finite] space. The numerous proofs which in the course of time were adduced in demonstration of these axioms, especially of the eleventh, all turn out on close examination to be pseudo-proofs. Legendre drew attention to the fact that either of the following axioms might be substituted for the eleventh:

a) Given a straight line, there can be drawn through a point in the same plane with that line, one and one line only which shall not intersect the first (parallels) however far the two lines may be produced;

b) If two parallel lines are cut by a third straight line, the interior alternate angles will be equal.

c) The sum of the angles of a triangle is equal to two right angles, that is, to the angle of a straight line or to 180°.

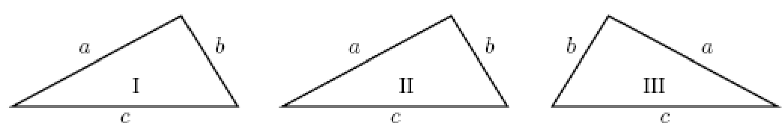


Fig. 1.

By the aid of any one of these three assertions, the eleventh axiom of Euclid may be proved, and, *vice versa*, by the aid of the latter each of the three assertions may be proved, of course with the help of the other two axioms, eight and twelve. The perception that the eleventh axiom does not admit of demonstration without the employment of one of the foregoing substitutes may best be gained from the consideration of congruent figures. Every reader will remember from his first instruction in geometry that the congruence of two triangles is demonstrated by the superposition of one triangle on the other and by then ascertaining whether the two completely coincide, no assumptions being made in the determination except those above mentioned.

In the case of triangles which are congruent, as are I and II in the preceding cut, this coincidence may be effected by the simple *displacement* of one of the triangles; so that even a two-dimensional being, supposed to be endowed with powers of reasoning, but only capable of picturing to itself motions within a plane, also might convince itself that the two triangles I and II could be made to coincide. But a being of this description could not convince itself in like manner of the congruence of triangles I and III. It would discover the equality of the three sides and the three angles, but it could never succeed in so superposing the two triangles on each other as to make them coincide. A three-dimensional being, however, can do this very easily. It has simply to turn triangle I about one of its sides and to shove the triangle, thus brought into the position of its reflexion in a mirror, into the position of triangle III. Similarly, triangles II and III may be made to coincide by moving either out of the plane of the paper around one of its sides as axis and turning it until it again falls in the plane of the paper. The triangle thus turned over can then be brought into the position of the other.

Later on we shall revert to these two kinds of congruence: "congruence by displacement" and "congruence by circumversion." For the present we will start from the fact that it is always possible within the limits of a plane to take a triangle out of one position and bring it into another without altering its sides and angles. The question is, whether this is only possible in the plane, or whether it can also be done on other surfaces.

We find that there are certain surfaces in which this is possible, and certain others in which it is not. For instance, it is impossible to move the triangle drawn on the surface of an egg into some other position on the egg's surface without a distension or contraction of some of the triangle's parts. On the other hand, it is quite possible to move the triangle drawn on the surface of a sphere into any other position on the sphere's surface without a distension or contraction of its parts. The mathematical reason of this fact is, that the surface of a sphere, like the plane, has everywhere the same curvature, but that the surface of an egg at different places has different curvatures. Of a plane we say that it has everywhere the curvature zero; of the surface of a sphere we say it has everywhere a positive curvature, which is greater in proportion as the radius is smaller. There are surfaces also which have a constant negative curvature; these surfaces exhibit at every point in directions proceeding from the same side a partly concave and a partly convex structure, somewhat like the centre of a saddle. There is no necessity of our entering in any detail into the character and structure of the last-mentioned surfaces.

Intimately related with the plane, however, are all those surfaces, which, like the plane, have the curvature zero; in this category belong especially cylindrical surfaces and conical surfaces. A sheet of paper of the form of the sector of a circle may, for example, be readily bent into the shape of a conical surface. If two congruent triangles, now, be drawn on the sheet of paper, which may by displacement be translated the one into the other, these triangles will, it is plain, also remain congruent on the conical surface; that is, on the conical surface also we may displace the one into the other; for though a bending of the figures will take place, there will be no distension or contraction. Similarly, there are surfaces which, like the sphere, have everywhere a constant positive curvature. On such surfaces also every figure can be transferred into some other position without distension or contraction of its parts. Accordingly, on all surfaces thus related to the plane or sphere, the assumption which underlies the eighth axiom of Euclid, that it is possible to transfer into any new position any figure drawn on such surfaces without distortion, holds good.

The eleventh axiom in its turn also holds good on all surfaces of constant curvature, whether the curvature be zero or positive; only in such instances instead of "straight" line we must say "shortest" line. On the surface of a sphere, namely, two shortest lines, that is, arcs of two great circles, always intersect, no matter whether they are produced in the direction of the side at which the third arc of a great circle makes with them angles less than two right angles, or, in the direction of the other side, where this arc makes with them angles of more than two right angles. On the plane, however, two straight lines intersect only on the side where a third straight line that meets them makes with them interior angles less than two right angles.

The twelfth axiom of Euclid, finally, only holds good on the plane and on the surfaces related to it, but not on the sphere or other surfaces which, like the sphere, have a constant positive curvature. This also accounts for the fact that one of the three postulates which we regarded as substitutes for the eleventh axiom, though valid for the plane, is not true for the surface of a sphere; namely, the postulate that defines the sum of the angles of a triangle. This sum in a plane triangle is two right angles; in a spherical triangle it is more than two right angles, the spherical triangle being greater, the greater the excess the sum of its angles is above two right angles. It will be seen, from these considerations, that in geometries in which curved surfaces and not fixed planes are studied, the axioms of Euclid are either all or partially false.

The axioms of geometry thus having been revealed as facts of experience, the question suggested itself whether in the same way in which it was shown that different two-dimensional geometries were possible, also different three-dimensional systems of geometry might not be developed; and consequently what the relations were in which these might stand to the geometry of the space given by our senses and representable to our mind. As a fact, a three-dimensional geometry can be developed, which like the geometry of the surface of an egg will exclude the axiom that a figure or body can be transferred from any one part of space to any other and yet remain congruent to itself. Of a three-dimensional space in which such a geometry can be developed we say, that it has no constant measure of curvature.

The space which is representable to us, and which we shall henceforth call the *space of experience*, possesses, as our experiences without exception confirm, the especial property that every bodily thing can be transferred from any one part of it to any other without suffering in the transference any distension or any contraction. The space of experience, therefore, has a constant measure of curvature. The question, however, whether this measure of curvature is zero or positive, that is, whether the space of experience possesses the properties which in two-dimensional structures a plane possesses, or whether it is the three dimensional analogue of the surface of a sphere is one which future experience alone can answer. If the space of experience has a constant positive measure of curvature which is different from zero, be the difference ever so slight, a point which should move forever onward in a straight line, or, more accurately expressed, in a shortest line, would sometime, though perhaps after having traversed a distance which to us is inconceivable, ultimately have to arrive from the opposite direction at the place from which it set out, just as a point which moves forever onward in the same direction on the surface of a sphere must ultimately arrive at its starting point, the distance it traverses being longer the greater the radius of the sphere or the smaller its curvature.

It will seem, at first blush, almost incredible, that the space of experience possibly could have this property. But an example, which is the historical analogue of this modern transformation of our conceptions, will render the idea less marvellous. Let us transport ourselves to the age of Homer. At that time people believed that the earth was a great disc surrounded on all sides by oceans which were conceived to be in all directions infinitely great. Indeed, for the primitive man, who has never journeyed far from the place of his birth, this is the most natural conception. But imagine now that some scholar had come, and had informed the Homeric hero Ulysses that if he would travel forever on the earth in the same direction he would ultimately come back to the point from which he started; surely Ulysses would have gazed with as much astonishment upon this scholar as we now look upon the mathematician who tells us that it is possible that a point which moves forever onward in space in the same direction may ultimately arrive at the place from which it started. But despite the fact that Ulysses would have regarded the assertion of the scholar as false because contradictory to his familiar conceptions, that scholar, nevertheless, would have been right; for the earth is not a plane but a spherical surface. So also the mathematician may be right who bases this more recent strange view on the possible fact that the space of experience may have a measure of curvature which is not exactly zero but slightly greater than zero. If this were really the case, the volume of the space of experience, though very large, would, nevertheless, be finite; just as the real spherical surface of the earth as contrasted with the Homeric plane surface is finite, having so and so many square miles. When the objection is here made that a finiteness of space is totally at variance with our modes of thought and conceptions, two ideas, "infinitely great" and "unlimited," are confounded. All that is at variance with our practical conceptions is that space can anywhere have a boundary; not that it may possibly be of tremendous but finite magnitude.

It will now be asked if we cannot determine by actual observation whether *the measure of curvature of experiential space is exactly zero or slightly different therefrom. The theorem of the sum of the angles of a triangle and the conclusions which follow from this theorem do indeed supply us with a means of ascertaining this fact. And the results of observation have been, that the measure of curvature of space is in all probability exactly equal to zero or if it is slightly different from zero it is so little so that the technical means of observation at our command and especially our telescopes are not competent to determine the amount of the deviation.* More, we cannot with any certainty say.

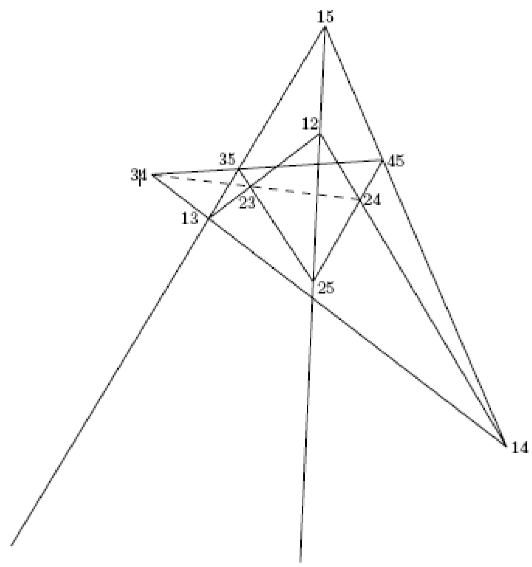


Fig. 2.

All these reflections, to which the criticism of the hypotheses that underlie geometry long ago led investigators, compel us to institute a comparison between the space of experience and other threedimensional aggregates of points (spaces), which we cannot mentally represent but can in thought and word accurately define and investigate. As soon, however, as we are fully implicated in the task of accurately investigating the properties of three-dimensional aggregates of points, we find ourselves similarly forced to regard such aggregates as the component elements of a manifoldness of more than three dimensions. In this way the exact criticism of even ordinary geometry leads us to the abstract assumption of a space of more than three dimensions. And as the extension of every idea gives a clearer and more translucent form to the idea as it originally stood, here too the idea of multi-dimensioned aggregates of points and the investigation of their properties has thrown a new light on the truths of ordinary geometry and placed its properties in clearer relief. Among the numerous examples which show how the notion of a space of multiple dimensions has been of great service to science in the investigation of three-dimensional space, we shall give one a place here which is within the comprehension of non-mathematicians.

Imagine in a plane two triangles whose angles are denoted by pairs of numbers-namely, by $1-2$, $1-3$, $1-4$, and $2-5$, $3-5$, $4-5$. (See Fig. 2.) Let the two triangles so lie that the three lines which join the angles $1-2$ and $2-5$, $1-3$ and $3-5$, and $1-4$ and $4-5$ intersect at a point, which we will call $1-5$. If now we cause the sides of the triangles which are opposite to these angles to intersect, it will be found that the points of intersection so obtained possess the peculiar property of lying all in one and the same straight line. The point of intersection of the connection $1-3$ and $1-4$ with the connection $4-5$ and $3-5$ may appropriately be called $3-4$. Similarly, the point of intersection $2-4$ is produced by the meeting of $4-5$, $2-5$ and $1-2$, $1-4$; and the point of intersection $2-3$, by the meeting of $1-3$, $1-2$ and $3-5$, $2-5$. The statement, that the three points of intersection $3-4$, $2-4$, $2-3$, thus obtained, lie in one straight line, can be proved by the principles of plane geometry only with difficulty and great circumstantiality. But by resorting to the threedimensional space of experience, in which the plane of the drawing lies, the proposition can be rendered almost self-evident.

To begin with, imagine any five points in space which may be denoted by the numbers 1, 2, 3, 4, 5; then imagine all the possible ten straight lines of junction drawn between each two of these points, namely, $1-2$, $1-3 \ldots 4-5$; and finally, also, all the ten planes of junction of every three points described, namely, the plane $1-2-3$, $1-2-4$, $\ldots 3-4-5$. A spatial figure will thus be obtained, whose ten straight lines will meet some interposed plane in ten points whose relative positions are exactly those of the ten points above described. Thus, for example, on this plane the points $1-2$, $1-3$, and $2-3$ will lie in a straight line, for through the three spatial points 1, 2, 3, a plane can be drawn which will cut the plane of a drawing in a straight line. The reason, therefore, that the three points $3-4$, $2-4$, $2-3$, also must ultimately lie in a straight line, consists in the simple fact that the plane of the three points 2, 3, 4, must cut the plane of the drawing in a straight line. The figure here considered consists of ten points of which sets of three so lie ten times in a straight line that conversely from every point also three straight lines proceed.

Now, just as this figure is a section of a complete three-dimensional pentagon, so another remarkable figure, of similar properties, may be obtained from the section of a figure of four-dimensioned space. Imagine six points, 1, 2, 3,

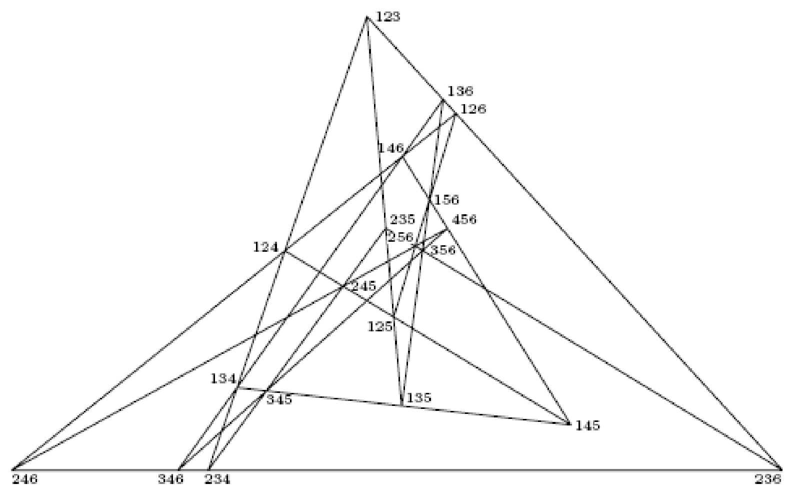


Fig. 3.

4, 5, 6, situated in this four-dimensioned space, and every three of them connected by a plane, and every four of them by a three-dimensioned space. We shall obtain thus twenty planes and fifteen three-dimensioned spaces which will cut the plane in which the figure is to be produced in twenty points and fifteen rays which so lie that each point sends out three rays and every ray contains four points. (See Fig. 3.) Figures of this description, which are so composed of points and rays that an equal number of rays proceed from every point and an equal number of points lie in every ray, are called *configurations*. Other configurations may, of course, be produced, by taking a different number of points and by assuming that the points taken lie in a space of different or even higher dimensions. The author of this article was the first to draw attention to configurations derived from spaces of higher dimensions. As we see, then, the notion of a space of more than three dimensions has performed an important service in the investigations of common plane geometry.

In conclusion, I should like to add a remark which Cranz makes regarding the application of the idea of multi-dimensioned space to theoretical chemistry. (See the treatise before cited.) In chemistry, the molecules of a compound body are said to consist of the atoms of the elements which are contained in the body, and these are supposed to be situated at certain distances from one another, and to be held in their relative positions by certain forces. At first, the centres of the atoms were conceived to lie in one and the same plane. But Wislicenus was led by researches in paralactic acid to explain the differences of isomeric molecules of the same structural formulæ by the different positions of the atoms in space. (Compare *La chimie dans l'espace* by van't Hoff, 1875, preface by J.

Wislicenus). In fact four points can always be so arranged in space that every two of them may have any distance from each other; and the change of one of the six distances does not necessarily involve the alteration of any other.

But suppose our molecule consists of five atoms? Four of these may be so placed that the distance between any two of them can be made what we please. But it is no longer possible to give the fifth atom a position such that each of the four distances by which it is separated from the other atoms may be what we please. On the contrary, the fourth distance is dependent on the three remaining distances; for the space of experience has only three dimensions. If, therefore, I have a molecule which consists of five atoms I cannot alter the distance between two of them without at least altering some second distance. But if we imagine the centres of the atoms placed in a four-dimensioned space, this can be done; all the ten distances which may be conceived to exist between the five points will then be independent of one another. To reach the same result in the case of six atoms we must assume a five-dimensional space; and so on.

Now, if the independence of all the possible distances between the atoms of a molecule is absolutely required by theoretical chemical research, the science is really compelled, if it deals with molecules of more than four atoms, to make use of the idea of a space of more than three dimensions. This idea is, in this case, simply an instrument of research, just as are, also, the ideas of molecules and atoms—means designed to embrace in a perspicuous and systematic form the phenomena of chemistry and to discover the conditions under which new phenomena can be evoked. Whether a four-dimensioned space really exists is a question whose insolubility cannot prevent research from making use of the idea, exactly as chemistry has not been prevented from making use of the notion of atom, although no one really knows whether the things we call atoms exist or not.

V. Refutation of the Arguments Adduced to prove the Existence of a Four-Dimensioned Space Inclusive of the Visible World.

The considerations of the preceding section will have convinced the cultured non-mathematician of the service which the theory of multidimensioned spaces has done, and bids fair to do, for geometrical research. In addition thereto is the consideration that every extension of one branch of mathematical science is a constant source of beneficial and helpful influence to the other branches. The knowledge, however, that mathematicians can employ the notion of four-dimensioned space with good results in their researches, would never have been sufficient to procure it its present popularity; for every man of intelligence has now heard of it, and, in jest or in earnest, often speaks of it. The knowledge of a four-dimensioned space did not reach the ears of cultured nonmathematicians until the consequences which the spiritualists fancied it was permissible to draw from this mathematical notion were publicly known. But it is a tremendous step from the four-dimensioned space of the mathematicians to the space from which the spirit-friends of the spiritualistic mediums entertain us with rappings, knockings, and bad English. Before taking this step we will first discuss the question of the real existence of a four-dimensional space, not deciding the question whether this space, if it really does exist, is inhabited by reasonable beings who consciously act upon the world in which we exist.

Among the reasons which are put forward to prove the existence of a four-dimensional space containing the world, the least reprehensible are those which are based on the existence of symmetry. We spoke above of two triangles in the same plane which have all their sides and angles congruent, but which cannot be made to coincide by simple displacement within the plane; but we saw that this coincidence could be effected by holding fast one side of one triangle and moving it out of its plane until it had been so far turned round that it fell back into its plane. Now something similar to this exists in space. Cut two figures, exactly like that of Fig. 4, out of a piece of paper, and turn the triangle *ABF* about the side *AB*, *ACE* about the side *AC*, *BCD* about the side *BC*, and in one figure above and in the other below; then in both cases the points *D*, *E*, *F* will meet at a point, because *AE* is equal to *AF*, *BF* is equal to *BD*, *CD* is equal to

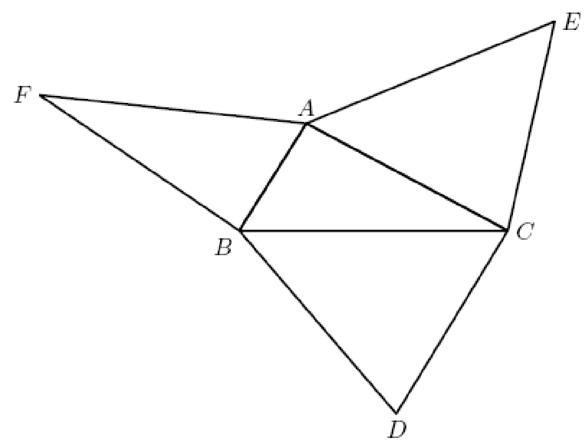


Fig. 4.

CE. In this manner we obtain two pyramids which in all lengths and all angles are congruent, yet which cannot, no matter how we try, be made to coincide, that is, be so fitted the one into the other that they shall both stand as one pyramid. But the *reflected* image of the one could be brought into coincidence with the other. Two spatial structures whose sides and angles are thus equal to each other, and of which each may be viewed as the reflected image of the other, are called *symmetrical*. For instance, the right and the left hand are symmetrical; or, a right and a left glove. Now just as in two dimensions it is impossible by simple displacement to bring into congruence triangles which like those above mentioned can only be made to coincide by circumversion, so also in three dimensions it is impossible to bring into congruence two symmetrical pyramids. Careful mathematical reflection, however, declares that this could be effected, if it were possible, while holding one of the surfaces, to move the pyramid out of the space of experience, and to turn it round through a four-dimensioned space until it reached a point at which it would return again into our experiential space. This process would simply be the four-dimensional analogue of the three-dimensional circumversion in the above-mentioned case of the two triangles. Further, the interior surfaces in this process would be converted into exterior surfaces, and vice versa, exactly as in the circumversion of a triangle the anterior and posterior sides are interchanged. If the structure which is to be converted into its symmetrical counterpart is made of a flexible material, the interchange mentioned of the interior and exterior surfaces may be effected by simply turning the structure inside out; for example, a right glove may thus be converted into a left glove.

Now from this truth, that every structure can be converted, by means of a four-dimensional space inclusive of the world, into a structure symmetrical with it, it has been sought to establish the probability of the real existence of a four-dimensioned space. Yet it will be evident, from the discussion of the preceding section, that the only inference which we can here make is, that the idea of a four-dimensioned space is competent, from a mathematical point of view, to throw some light upon the phenomena of symmetry. To conclude from these facts that a space of this kind really exists, would be as daring as to conclude from the fact that the uniform angular velocity of the apparent motions of the fixed stars is explicable from the assumption of an axial motion of the firmament, that the fixed stars are really rigidly placed in a celestial sphere rotating about its axis. It must not be forgotten that our comprehension of the phenomena of the real world consists of two elements: first, of that which the things really are; and, second, of that by which we rationally apprehend the things. This latter element is partly dependent on the sum of the experiences which we have before acquired, and partly on the necessity, due to the imperfection of reason, of our classifying the multitudinous isolated phenomena of the world into categories which we ourselves have formed, and which, therefore, are not wholly derived from the phenomena themselves, but to a great extent are dependent on us.

Besides geometrical reasons, Zöllner has also adduced cosmological reasons to prove the existence of a four-dimensional space. To these reasons belong especially the questions, whether the number of the fixed stars is infinitely great, whether the world is finite or infinite in extension, whether the world had a beginning or will have an end, whether the world is not hastening towards a condition of equilibrium or dead level by the universal distribution of its matter and energy; the problems, also, of gravitation and action at a distance; and finally, the questions concerning the relations between the phenomena in the world of sense-perception to the unknown things-in-themselves. All these questions which can be decided in no definite sense, led Zöllner and his followers to the assumption that a four-dimensioned space inclusive of the space of experience must really exist. But more careful reflection will show that this assumption does not dispose of the difficulties but simply displaces them into another realm. Furthermore, even if four-dimensioned space did unravel and make clear all the cosmological problems which have bothered the human mind, still, its existence would not be proved thereby; it would yet remain a mere hypothesis, designed to render more intelligible to a being who can only make experiences in a three-dimensional space, the phenomena therein which are full of mystery to it. A four-dimensioned space would in such case possess for the metaphysician a value similar to that which the ether possesses for the physicist. Still more convincing than these cosmological reasons to the majority of men is the physio-psychological reason drawn from the phenomena of vision which Zöllner adduces. Into this main argument we will enter in more detail.

When we “see” an object, as we all know, the light which proceeds or is reflected therefrom produces an image on the retina of our eye; this image is conducted to our consciousness by means of the optic nerve, and our reason draws therefrom an inference respecting the object. When, now, we look at a square whose sides are a decimetre in length, and whose centre is situated at the distance of a metre from the pupil of our eye, an image is produced on the retina. But exactly the same image will be produced there if we look at a square whose sides are parallel to the sides of the first square but two decimetres in length, and whose centre is situated at a distance of two metres from the pupil of the eye. Proceeding thus further, we readily discover that an eye can perceive in any length or line only the ratio of its magnitude to the distance at which it is situated from it, and that generally a threedimensional world must appear to the eye two-dimensional, because all points which lie behind each other in the direction outwards from the eye produce on the retina only one image. This is due to the fact that the retinal images are themselves two-dimensional; for which reason, Zöllner says, the world must appear to a child as two-dimensional, if it be supposed to live in a primitive condition of unconscious mental activity. To such a child two objects which are moving the one behind the other, must appear as suffering displacement on a surface, which we conceive behind the objects, and on which the latter are projected. In all these apparent displacements, coincidences and changes of form also are effected. All these things must appear puzzling to a human being in the first stages of its development, and the mind thus finds itself, as Zöllner further argues, in the first years of childhood forced to adopt a hypothesis concerning the constitution of space and to assume that the world is three-dimensional, although the eye can really perceive it as only two-dimensional. Zöllner then further says, that in the explanation of the effects of the external world, man constantly finds this hypothesis of his childish years confirmed, and that in this way it has become in his mind so profound a conviction that it is no longer possible for him to think it away. Consonant with this argumentation, also, is Zöllner’s remark, that the same phenomenon has presented itself in astronomical methods of knowledge. To explain the movements of the planets, which appear to describe regular paths on the surface of a celestial sphere, we were compelled in the solution of the riddles which these motions presented, to assume in the structure of the heavens a dimension of “depth,” and the complicated motions in the two-dimensioned firmament were converted into very simple motions in three-dimensioned space. Zöllner also contends that our conception of the entire visible world as possessed of three dimensions is a product of our reason, which the mind was driven to form by the contradictions which would be presented to it on the assumption of only two dimensions by the perspective distortions, coincidences, and changes of magnitude of objects. When a child moves its hand before its eyes, turns it, brings it nearer, or pushes it farther away, this child successively receives the most various impressions on the surface of its retina of one and the same object of whose identity and constancy its feelings offer it a perfect assurance. If the child regarded the changeable projection of the hand on the surface of the retina as the real object, and not the hand which lies beyond it, the child would constantly be met with contradictions in its experience, and to avoid this it makes the hypothesis that the space of experience is three-dimensional. Zöllner's contention is, therefore, that man originally had only a two-dimensional intuition of space, but was forced by experience to represent to himself the objects which on the retinal surface appeared two-dimensional, as three-dimensional, and thus to transform his two-dimensional space-intuition into a three-dimensional one. Now, in exactly the same way, according to Zöllner's notion, will man, by the advancement and increasing exactness of his knowledge of the phenomena of the outer world, also be compelled to conceive of the material world as a "shadow cast by a more real four-dimensional world," so that these conceptions will be just as trivial for the people of the twentieth century as since Copernicus's time the explanation of the motions of the heavenly bodies by means of a three-dimensional motion has been.

Zöllner's arguments from the phenomena of vision may be refuted as follows: In the first place it is incorrect to say that we see the things of the external world by means of two-dimensional retinal images. The light which penetrates the eye causes an irritation of the optic nerves, and any such effect which, though it be not powerful, is, nevertheless, a mechanical one, can only take place on things which are material. But material things are always three-dimensional. The effect of light on the sensitive plates of photography can with just as little justice be regarded as two-dimensional. Our senses can have perception of nothing but three-dimensional things, and this perception is effected by forces which in their turn act on three-dimensional things, namely our sensory nerves. It is wrong to call an image two-dimensional, for it is only by abstraction that we can conceive of a thickness so growing constantly smaller and smaller as to admit of our regarding a three-dimensional picture as two-dimensional, by giving it in mind a vanishingly small thickness. It is also wrong to say, as Zöllner says, that when we see the shadow of a hand which is cast upon a wall we see something twodimensional. What we really perceive is that no light falls upon our eye from the region included by the shadow, while from the entire surrounding region light does fall on our eye. But this light is reflected from the material particles which form the surface of the wall, that is, from three-dimensional particles of matter. We must always remember that our eye communicates to us only three-dimensional knowledge, and that for the comprehension of anything which has two, one, or no dimensions, a *purely intellectual act of abstraction must be added to the act of perception*. When we imagine we have made a lead-pencil mark on paper, we have, exactly viewed, simply heaped alongside of each other little particles of graphite in such a manner that there are by far fewer graphite particles in the lateral and upward directions than there are in the longitudinal direction, and thus our reason arrives by abstraction at the notion of a straight line. When we look at an object, say a cube of wood, we recognise the object as three-dimensional, and it is only by abstraction that we can conceive of its two-dimensional surfaces, of its twelve one-dimensional edges, and of its eight no-dimensional corners. For we reach the perception of its surface, for example, solely by reason of the fact that the material particles which form the cube prevent the transmission of light, and reflect it, whereby a part of the light reflected from every material particle strikes our eye. Now, by thinking exclusively of those material particles which are reflected, in contrariety to the empty space without and the hidden and therefore non-reflected particles within, we form the notion of a surface.

It is evident from this, that all that we perceive is three-dimensional, that we cannot reach anything two-dimensional without an intellectual abstraction, and that, therefore, we cannot conceive of anything two-dimensional exerting effects upon material things. But this fact is a refutation of the retinal argument of Zöllner. If vision consisted wholly and exclusively in the creation of a two-dimensional image, the things which take place in the world could never come into our consciousness. The child, therefore, does not originally apprehend the world, as Zöllner says, as two-dimensional; on the contrary, it apprehends it either not at all, or it apprehends it as three-dimensional. Of course the child must first "learn how" to see. It is found from the observation of children during the first months of their lives, and of the congenitally blind who have suddenly acquired the power of vision by some successful operation, that seeing does not consist alone in the irritations which arise in the optic nerves, but also in the correct interpretation of these irritations by reason. This correct interpretation, however, can be accomplished only by the accumulation of a considerable stock of experience. Especially must the recognition of the distance of the object seen be gradually learned. In this, two things are especially helpful; first, the fact that we have two eyes and, consequently, that we must feel two irritations of the optic nerves which are not wholly alike; and, secondly, the fact that we are enabled by our power of motion and our sense of touch to convince ourselves of the distance and form of the bodies seen. The question now arises, what sort of an intuition of space would a creature have that had only one eye, that could neither move itself nor its eye, and also possessed no peripheral nerves. According to Zöllner's view, this creature could, owing to its two-dimensional retinal images, have only a two-dimensional intuition of space. The author's opinion, however, is, that such a creature could not see at all, as it has no possibility of collecting experiences which are adapted in any way to interpreting the effects of things on its retina. The light which proceeded from the objects roundabout and fell on the retina could produce no other effect on the being than that of a wholly unintelligible irritation, or perhaps even pain.

The reflections presented sufficiently show that neither the phenomena of symmetry nor the retinal images of the objects of vision necessarily force upon us the assumption of a four-dimensioned space. If the material world should ever present problems which could not in the progress of knowledge be solved in a natural way, the assumption that a four-dimensional space containing the world exists would also be incompetent to resolve the difficulties presented; it would simply convert these difficulties into others, and not dispose of the problems but simply displace them to another world. Yet the question might be asked, is the existence of a four-dimensional space really *impossible?* To answer this question we must first clearly know what we mean by "exist." If existence means that the intellectual idea of a thing can be formed and that this idea shall not lead to contradictions with other well-established ideas and with experience, we have only to say that four-dimensioned space does exist, as the arguments adduced in sections III and IV have rendered plain. If, namely, the space of four dimensions did not exist as a clear idea in the minds of mathematicians, mathematicians could certainly not have been led by this idea to results which are recognised by the senses as true, and which really take place in our own representable space. But if existence means "material actuality," we must say that we neither now nor in the future can know anything about it. For we know material actuality only as three-dimensional, our senses can only make three-dimensional experiences, and the inferences of our reason, although they can well abstract from material things, can never ascend to the point of explaining a four-dimensional materiality. Just as little, therefore, as we can locally fix the idea of a two-dimensional material world, as little can we ever verify the notion of a four-dimensional material existence.

VI. Examination of the Hypothesis Concerning the Existence of Four-Dimensional Spirits.

In connection with the belief that the visible world is contained in a four-dimensioned space, Zöllner and his adherents further hold that this higher space is inhabited by intelligent beings who can act consciously and at will on the human beings who live in experiential space. To invest this opinion with greater strength, Zöllner appealed to the fact that the greatest thinkers of antiquity and of modern times were either wholly of this opinion or at least held views from which his contentions might be immediately derived. Plato's dialogue between Socrates and Glaukon in the seventh book of the Republic, is evidence, says Zöllner, that this greatest philosopher of antiquity possessed some presentiment of this extension of the notion of space. Yet any one who has connectedly studied and understood Plato's system of philosophy must concede that the so-called "ideas" of the Platonic system denote something wholly different from what Zöllner sees in them or pretends to see. Zöllner says that these Platonic ideas are spatial objects of more than three dimensions and represent "real existence" in the same sense that the material world, as contrasted with the images on the retina, represents it. Zöllner similarly deals with the Kantian "thing-in-itself," which is also regarded as an object of higher dimensions.

To show Kant in the light of a predecessor, Zöllner quotes the following passage from the former's "Tr¨aume eines Geistersehers, erl¨autert durch Tr¨aume der Metaphysik" (1766, *Collected Works*, Vol. VII. page 32 et seq.): "I confess that I am very much inclined to assert the existence of immaterial beings in the world, and to rank my own soul as one of such a class. It appears, there is a spiritual essence existent which is intimately bound up with matter but which does not act on those forces of the elements by which the latter are connected, but upon some internal principle of its own condition. It will, in the time to come—I know not when or where—be proved, that the human soul, even in this life, exists in a state of uninterrupted connection with all the immaterial natures of the spiritual world; that it alternately acts on them and receives impressions from them, of which, as a human soul, it is not, in the normal state of things, conscious. It would be a great thing, if some such systematic constitution of the spiritual world, as we conceive it, could be deduced, not exclusively from our general notion of spiritual nature, which is altogether too hypothetical, but from some real and universally admitted observations,—or, for that matter, if it could even be shown to be probable."

What Kant really asserts here is, first, the partly independent and partly dependent existence of the soul, and of spiritual beings generally, on matter, and, second, that spiritual beings have some common connection with and mutually influence one another. This contention, which is that of very many thinkers, does not, however, entail the consequence that the "transcendental subject of Kant" must be fourdimensional, as Zöllner asserts it does. Kant never even hinted at the theory that the psychical features of the world owe their connection with the material features to the fact that they are four-dimensional and, therefore, include the three-dimensional. Is it a necessary conclusion that if a thing exists and is not three-dimensional, as is the case with the soul, it is therefore four-dimensional? Can it not in fact be so constituted that it is wholly meaningless to speak of dimensions at all in connection with it?

Yet still more strangely than the words of Plato and Kant do certain utterances of the mathematicians Gauss and Riemann speak in favor of Zöllner's hypothesis. S. v. Waltershausen relates of Gauss in his *Gruss zum Gedächtniss*, (Leipsic, 1856), that Gauss had often remarked that the three dimensions of space were only a specific peculiarity of the human mind. We can think ourselves, he said, into beings who are conscious of only two dimensions; similarly, perhaps, beings who are above and outside our world may look down upon us; and there were, he continued, in a jesting tone, a number of problems which he had here indefinitely laid aside, but hoped to treat in a superior state by superior geometrical methods. Leaving aside this jest, which quite naturally suggested itself, the remarks of Gauss are quite correct. We possess the power to abstract and can think, therefore, what kind of geometry a being that is only acquainted with a two-dimensional world would have; for instance, we can imagine that such a being could not conceive of the possibility of making two triangles coincide which were congruent in the sense above explained, and so on. So, also, we can understand that a being who has control of four dimensions can only conceive of a geometry of four-dimensional space, yet may have the capacity of thinking itself into spaces of other dimensions. But it does not follow from this that a four-dimensional space exists, let alone that it is inhabited by reasonable beings.

Riemann, on the other hand, speaks directly of a world of spirits. In his *Neue mathematische Principien der Naturphilosophie* he puts forth the hypothesis that the space of the world is filled with a material that is constantly pouring into the ponderable atoms, there to disappear from the phenomenal world. In every ponderable atom, he says, at every moment of time, there enters and appears a determinate amount of matter, proportional to the force of gravitation. The ponderable bodies, according to this theory, are the place at which the spiritual world enters and acts on the material world. Riemann's world of spirits, the sole office of which is to explain the phenomenon of gravitation as a force governing matter, is, however, essentially different from the spiritual world of Zöllner, the function of which is to explain supposed supersensuous phenomena which stand in the most glaring contradiction with the established known laws of the material world.

Besides this appeal to the testimony of eminent men like Plato, Kant, Gauss, and Riemann, the scientific prophet of modern spiritualism also bases his theory on the belief, which has obtained at all times and appeared in various forms among all peoples, that there exist in the world forces which at times are competent to evoke phenomena that are exempt from the ordinary laws of nature. We have but to think of the phenomena of table-turning which once excited the Chinese as much as it has aroused, during the last few decades, the European and American worlds; or of the divining-rod, by whose help our forefathers sought for water, in fact, as we do now in parts of Europe and America.

Cranz, in his essay on the subject, divides spiritualistic phenomena into physical and intellectual. Of the first class he enumerates the following: the moving of chairs and tables; the animation of walkingsticks, slippers, and broomsticks; the miraculous throwing of objects; spirit-rappings (Luther heard a sound in the Wartburg, "as if three score casks were hurled down the stairs"); the ecstatic suspension of persons above the floor; the diminution of the forces of gravity; the ordeals of witches; the fetching of wished-for objects; the declination of the magnetic needle by persons at a distance; the untying of knots in a closed string; insensibility to injury and exemption therefrom when tortured, as in handling red-hot coals, carrying hot irons, etc.; the music of invisible spirits; the materialisations of spirits or of individual parts of spirits (the footprints in the experiments of Slade, photographed by Zöllner); the double appearance of the same person; the penetration of matter (of closed doors, windows, and so forth). As numerous also is the selection presented by Cranz of intellectual phenomena, namely: spirit-writing (Have's instrument for the facilitation of intercourse with spirits), the clairvoyance and divination of somnambulists, of visionary, ecstatic, and hypnotised persons, prompted or controlled by narcotic medicines, by sleeping in temples, by music and dancing, by ascetic modes of life and residence in barren localities, by the exudations of the soil and of water, by the contemplation of jewels, mirrors, and crystal-pure water, and by anointing the finger-nails with consecrated oil. Also the following additional intellectual phenomena are cited: increased eloquence or suddenly acquired power of speaking in foreign languages; spirit-effects at a distance; inability to move, transferences of the will, and so forth.

All these phenomena, presented with the aspect of truth, and associated more or less with trickery, self-deception, and humbug, are adduced by the spiritualists to substantiate the belief in a world of spirits which intentionally and consciously take part in the events of the material world, and to prove that these phenomena may be sufficiently and consistently explained by the effects of the activity of such a world. It is impossible for us to discuss and put to the test here the explanations of all these supersensuous phenomena. Anything and everything can be explained by spirits who act at will upon the world. There are only a few of these phenomena, namely, clairvoyance and Slade's experiments, whose explanations are so intimately connected with our main theme, the so-called fourth dimension, that they cannot be passed over.

First, with respect to clairvoyance, the American visionary Davis describes the experiences which he claims to have made in this condition, induced by "magnetic sleep," as follows:[7] "The sphere of my vision now began to expand. At first, I could only clearly discern the walls of the house. At the start they seemed to me dark and gloomy; but they soon became brighter and finally transparent. I could now see the objects, the utensils, and the persons in the adjoining house as easily as those in the room in which I sat. But my perceptions extended further still; before my wandering glance, which seemed to control a great semi-circle, the broad surface of the earth, for hundreds of miles about me, grew as transparent as water, and I saw the brains, the entrails, and the entire anatomy of the beasts that wandered about in the forests of the Eastern Hemisphere, hundreds and thousands of miles from the room in which I sat." The belief in the possibility of such states of clairvoyance is by no means new. Alexander Dumas made use of it, for example, in his *Méemoires d'un méedicin*, in which Count Balsamo, afterwards called Cagliostro, is said to possess the power to throw suitable persons into this wonderful condition and thus to find out what other persons at distant places are doing. Zöllner explains clairvoyance by means of the fourth dimension thus:

A man who is accustomed to viewing things on a plain is supposed to ascend to a considerable height in a balloon. He will there enjoy a much more extended prospect than if he had remained on the plain below, and will also be able to signal to greater distances. The plain, that is, the two-dimensioned space, is accordingly viewed by him from points outside of the plain as "open" in all directions. Exactly so, in Zöllner's theory, must three-dimensioned spaces appear, when viewed from points in four-dimensioned space, namely, as "open"; and the more so in proportion as the point in question is situated at a greater distance from the place of our body, or in proportion as the soul ascends to a greater height in this fourth dimension. Zöllner thus explains clairvoyance as a condition in which the soul has displaced itself out of its three-dimensioned space and reached a point which with respect to this space is four-dimensionally

[7] Quoted by Cranz.

situated and whence it is able to contemplate the three-dimensional world without the interference of intervening obstacles.

The following remark is to be made to this explanation. The reason why we have a better and more extended view from a balloon than from places on the earth is simply this, that between the suspended balloon and the objects seen at a distance nothing intervenes but the air, and air allows the transmission of light, whereas, at the places below on the earth there are all kinds of material things about the observer which prevent the transmission of light and either render difficult or absolutely impossible the sight of things which lie far away. In the same way, also, from a point in four-dimensioned space, a three-dimensional object will be visible only provided there are no obstacles intervening. If, therefore, this awareness of a distant object is a real, actual vision by means of a luminous ray which strikes the eye, there is contained in the explanation of Zöllner the tacit assumption that the medium with which the four-dimensional world is filled is also pervious to light exactly as the atmosphere is.

The theory that there are four-dimensional spirits who produce the phenomena cited by the spiritualists received special support from the experiments which the prestidigitateur Slade, who claimed he was a spiritualistic medium, performed before Zöllner. Of these experiments we will speak of the two most important, the experiment with the glass sphere and the experiment with the knots. To explain the connection of the glass sphere experiment with the fourth dimension, we must first conceive of two-dimensional reasoning beings, or, let us say, two-dimensional worms, living and moving in a plane. For a creature of this kind it will be self-evident that there are no other paths between two points of its plane than such as lie within the plane. It must, accordingly, be beyond the range of conception of this worm, how any two-dimensional object which lies within a circle in its space can be brought to any other position in its space outside the circle without the object passing through the barriers formed by the circle's circumference. But if this worm could procure the services of a three-dimensional being, the transportation of the object from a position within the circle to any position outside it could be effected by the three-dimensional being simply taking the object *out* of the plane and placing it at the desired point. This object, therefore, would, in an inexplicable manner, suddenly disappear before the eyes of the worms who were assembled as spectators, and after the lapse of an interval of time would again appear outside the circle without having passed at any point through the circle's circumference. If now we add another dimension, we shall derive from this trick, which is wholly removed from the sense-perception of the flattened worms, the following experiment, which is wholly beyond the perception of us human beings. Inside a glass sphere, which is closed all around, a grain of corn is placed; the problem is to transport the corn to some place outside the sphere without passing through the glass surface. Now we should be able to perform this trick if some four-dimensional being would render us the same aid that we before rendered the two-dimensional worm. For the four-dimensional being could take the grain of corn into his four-dimensional space and then replace it in our space in the desired spot outside of the glass sphere. Slade performed this trick before Zöllner. Its mere performance sufficed to convince this latter investigator that Slade had here made use of a four-dimensional agent, who, in respect of power of motion, controlled his four-dimensional space as we do our three-dimensional space. It never occurred to Zöllner that this experiment was the cleverly executed trick of a prestidigitateur, or, as it would at once occur to us, that the whole thing was a sensory illusion. The fact that we cannot explain a trick easily and naturally does not irrevocably prove that it is accomplished by other means than those which the world of matter presents.

Still better known than this last performance is Slade's experiments with knots. To explain this in connection with the fourth dimension, we must resort again to the plane and the flat worm inhabiting it. To two parallel lines in a plane let the two ends of a third line, which has a double point, that is, intersects itself once, be attached. Our flat worm would not be able to untie the loop formed by the doubled third line, which we will call a string, because it cannot execute motions in three dimensions. If, therefore, a two-dimensional prestidigitateur should appear and accomplish the trick of untying this loop without removing the two ends of the string from the parallel lines, he will have accomplished for our flat worm a supersensuous experiment. A human being engaged in the service of the prestidigitateur could execute for him the experiment by simply lifting the string a little out of the plane, pulling it taut, and placing it back again. This suggests the following analogous experiment for three-dimensional beings. The two ends of a string in which a common knot has been made are sealed to the opposite walls of a room. The problem is to untie this knot without breaking the seals at the two ends of the string. Everybody knows that this problem is not soluble, but it may be calculated mathematically that the knot in the string can be untied as easily by motions in a fourth dimension of space as in the experiment above described the knot in the two-dimensional string was untied by a three-dimensional motion. Now as Slade untied the knot before Zöllner's eyes without apparently making any use of the ends fastened in the walls, Zöllner was still more firmly confirmed in the view that Slade had power over spirits who performed the experiments for him.

Still more far-reaching is the theory of Carl du Prel concerning the relations of the material and the four-dimensional world. (Compare his numerous essays in the spiritualistic magazine *Sphinx*.) Just as the shadows of three-dimensional objects cast on a wall are controlled in their movements by the things whose projections they are, in the same way it is claimed does there exist back of everything of this senseperceptible world a real transcendental and four-dimensional “thing-initself” whose projection in the space of experience is what we falsely regard as the independent thing. Thus every man besides existing in his terrestrial self also exists in a spiritual or astral self which constantly accompanies him in his walks through life and whose existence is especially proclaimed in states of profound sleep, of somnambulism, and in the conditions of mediums. In this way Du Prel explains the wonderful feats of somnambulists, and the aerial journeys of sorcerers and witches. Whereas, ordinarily the separation of the material body from the astral body is only effected at the moment of death; in the case of somnambulists this separation may take place at any time, or, as Du Prel says, “the threshold of feeling may be permanently displaced.”

In view of the natural relations of such theories to the dogmas of Christianity it is explainable that theologians also have raised their voices for or against spiritualism. While the *Protestant Church Times* beheld in the “repulsive thaumaturgic performances which these coryphæi of modern science offer, a lack of comprehension of real philosophy,” the magazine *The Proof of Faith* expresses its delight at the discovery of spiritualism in the following manner: “Every Christian will surely rejoice at the deep and perhaps mortal wound which these new discoveries have in all probability administered to modern materialism.”

We shall pass by the childish opinion that the Bible also speaks of four dimensions, as both in Job (xi. 8-9) and in the Epistle to the Ephesians (iii. 18) only breadth, length, depth, and height, that is, four directions of extension, are mentioned. Yet we will still add, as Cranz has done, the reflections which Zöllner, as the most prominent representative of modern spiritualism, has put forward respecting its relations to the doctrines of Christianity (*Wissensch. Abhandl.*, Vol. III). By the foundation of transcendental physics on the basis of spiritualistic phenomena, the “new light” has arisen which is spoken of in the New Testament. The rending of the veil of the Temple on the crucifixion of Christ, the resurrection, the ascension, the transfiguration, the speaking with many tongues on the giving out of the Holy Ghost, all these are in Zöllner’s view spiritualistic phenomena. Similarly, he sees a reference to the four-dimensional world of spirits in all those sayings of Christ in which the latter speaks to his Apostles of the impossibility of their having any image or notion of the place to which when he disappeared he would go and whence he would return. (Gospel of St. John, xii. 33, 36; xiv. 2, 3, 28; xvi. 5, 13.)

Ulrici, however, goes farthest in the mingling of spiritualistic and Christian beliefs; for he sees in the doctrine of spiritualism a means of strengthening belief in a supreme moral world-order and in the immortality of the soul. In answer to Ulrici’s tract ”Spiritualism So-called, a Question of Science” (1889)

Wundt wrote an annihilating reply bearing the title "Spiritualism, a Question of Science So-called." Wundt criticises the future condition of our soul according to spiritualistic hypotheses in the following sarcastic yet pertinent words, which Cranz also quotes: "(1) Physically, the souls of the dead come into the thraldom of certain living beings who are called mediums. These mediums are, for the present at least, a not widely diffused class, and they appear to be almost exclusively Americans. At the command of these mediums, departed souls perform mechanical feats which possess throughout the character of absolute aimlessness. They rap, they lift tables and chairs, they move beds, they play on the harmonica, and do other similar things. (2) Intellectually, the souls of the dead enter a condition which, if we are to judge from the productions which they deposit on the slates of the mediums, must be termed a very lamentable one. These slate-writings belong throughout in the category of imbecility; they are totally bereft of any contents. (3) The most favored, apparently, is the moral condition of the soul. According to the testimony which we have, its character cannot be said to be anything else than that of harmlessness. From brutal performances, such, for instance, as the destruction of bed-canopies, the spirits most politely refrain." Wundt then laments the demoralising effect which spiritualism exercises on people who have hitherto devoted their powers to some serious pursuit or even to the service of science. In fact it is a presumptuous and flagrant procedure to forsake the path of exact research, which slow as it is, yet always leads to a sure extension of knowledge, in the hope that some aimless, foolhardy venture into the realm of uncertainty will carry us farther than the path of slow toil, and that we can ever thus easily lift the veil which hides from man the problems of the world that are yet unsolved.

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Now that we have presented the opinions of others respecting the existence of a four-dimensional world of spirits, the author would like to develop one or two ideas of his own on the subject. In the preceding section it was stated that everything that we perceive by our senses is three-dimensional and that everything that possesses four or more dimensions can only be regarded as abstractions or fictions which our reason forms in its constant efforts after an extension and generalisation of knowledge. To speak of two-dimensional matter is as self-contradictory as the notion of four-dimensional matter. But a two or a four-dimensional world might exist in some other manner than a material manner, and for all we know in one which to us does not admit of representation. But in such a case, if it were without the power of affecting the material world, we should never be able to acquire any knowledge concerning its existence, and it would be totally indifferent to the people of the three-dimensional world, whether such a world existed or not. Just as an artist during his lifetime produces a number of different works of art, so also the Creator might have created a number of different-dimensioned worlds which in no wise interfere with one another. In such a case, any one world would not be able to know anything of any other, and we must consequently regard the question whether a four-dimensional world exists which is incapable of affecting ours, as insoluble. We have only to examine, therefore, the question whether the phenomenal world perhaps is a single individual in a great layer of worlds of which every successive one has one more dimension than the preceding and which are so connected with one another that each successive world contains and includes the preceding world, and, therefore, can produce effects in it. For our reason, which draws its inferences from the phenomena of this world, tells us, that if outside the three-dimensional world there exists a second fourdimensional world, containing ours, there is no reason why worlds of more or less dimensions should not, with the same right, also exist. But if now, as Zöllner and his adherents maintain, four-dimensional spirits exist which can act by the mere efforts of their own wills on our world, there is surely no reason why we three-dimensional beings should not also be able to produce effects on some two-dimensional animated world. Whether we have, generally, any such influence we do not know, but we certainly do know that we do not act purposely and consciously on a two-dimensional world. If, therefore, Zöllner were right, the plan of creation would possess the wonderful feature that four-dimensional beings are capable of arbitrarily affecting the threedimensional world, but that three-dimensional beings have no right in their turn consciously to affect a two-dimensional world.

The supporters of Zöllner's hypothesis will perhaps reply to the objection just made, that the plan of creation might, after all, possibly possess this wonderful peculiarity, that we human beings perhaps, in some higher condition of culture, will be able to act consciously on two-dimensional worlds, and that at any rate it is simply an inference by analogy to conclude from the non-existence of a relation between three and two dimensions that the same relation is also wanting in the case of four and three dimensions. As a matter of fact, the objection above made is not intended to refute Zöllner's hypothesis, but only to stamp it as very improbable. But despite this improbability Zöllner would still be right if the phenomena of the material world actually made his hypothesis necessary. That, however, is by no means the case. Although most of the phenomena to which the spiritualists appeal are probably founded on sense-illusions, humbug, and self-deception, it cannot be denied that there possibly do exist phenomena which cannot be brought into harmony with the natural laws now known. There always have been mysterious phenomena, and there always will be. Yet, as we have often seen that the progress of science has again and again revealed as natural what former generations held to be supernatural, it is certainly wholly wrong to bring in for the explanation of phenomena which now seem mysterious an hypothesis like that of Zöllner, by which everything in the world can be explained. If we adopt a point of view which regards it as natural for spirits arbitrarily to interfere in the workings of the world, all scientific investigation will cease, for we could never more trust or rely upon any chemical or physical experiment, or any botanical or zo¨ological culture. If the spirits are the authors of the phenomena that are mysterious to us, why should they also not have control of the phenomena which are not mysterious? The existence of mysterious phenomena justifies in no manner or form the assumption that spirits exist which produce them. Would it not be much simpler, if we *must* have supernatural influences, to adopt the na¨ýve religious point of view, according to which everything that happens is traceable to the direct, actual, and personal interference of a single being which we call God? Things which formerly stood beyond the sphere of our knowledge and were regarded as marvellous events, as a storm, for example, now stand in the most intimate connection with known natural laws. Things that formerly were mysterious are so no longer. If one hundred and fifty years ago some scientists were in the possession of our present knowledge of inductional electricity and had connected Paris and Berlin with a wire by whose aid one could clearly interpret in Berlin what another person had at that very moment said in Paris, people would have regarded this phenomenon as supernatural and assumed that the originator of this long-distance speaking was in league with spirits.

We recognise, thus, that the things which are termed supernatural depend to a great extent on the stage of culture which humanity has reached. Things which now appear to us mysterious, may, in a very few decades, be recognised as quite natural. This knowledge, however, is not to be obtained by the lazy assumption of bands of spirits as the authors of mysterious phenomena, but by performing what in physics and chemistry is called experiment. But the first and essential condition of all scientific experimenting is that the experimenter shall be absolutely master of the conditions under which the experiment is or is not to succeed. Now, this criterion of scientific experimenting is totally lacking in all spiritualistic experiments. We can never assign in their case the conditions under which they will or will not succeed. When all the preparations in a spiritualistic *séance* have been properly made, but nothing takes place, the beautiful excuse is always forthcoming that the "spirits were not willing," that there were "too many incredulous persons present," and so forth. Fortunately, in physical experiments these pretexts are not necessary. By the path of experiment, and not by that of transcendental speculation, physics has thus made incredible progress and has piled new knowledge strata on strata upon the old. Accordingly, the prospect is left that the mysteries which the conditions and properties of the human soul still present can be solved more and more by the methods of scientific experiment. To this end, however, it is especially necessary that the physio-psychological experiments in question should only be performed by men who possess the critical eye of inquiry, who are free from the dangers of self-illusion, and who are competent to keep apart from their experiments all superstition and deception. The history of natural science clearly teaches that it is only by this road that man can arrive at certain and well-established knowledge. If, therefore, there really is behind such phenomena as mind-reading, telepathy, and similar psychical phenomena, something besides humbug and self-illusion, what we have to do is to study privately and carefully by serious experiments the success or non-success of such phenomena, and not allow ourselves to be influenced by the public and dramatic performances of psychical artists, like Cumberland and his ilk.

The high eminence on which the knowledge and civilisation of humanity now stands was not reached by the thoughtless employment of fanciful ideas, nor by recourse to four-dimensional worlds, but by hard, serious labor, and slow, unceasing research. Let all men of science, therefore, band themselves together and oppose a solid front to methods that explain everything that is now mysterious to us by the interference of independent spirits. For these methods, owing to the fact that they can explain everything, explain nothing, and thus oppose dangerous obstacles to the progress of real research, to which we owe the beautiful temple of modern knowledge.